Control Systems on Lie Groups

P.S. Krishnaprasad
Department of Electrical and Computer Engineering
and
Institute for Systems Research
University of Maryland, College Park, MD 20742

Control Systems on Lie Groups

- Control systems with state evolving on a (matrix) Lie group arise frequently in physical problems.

- Initial motivation came from the study of bilinear control systems (control multiplies state).

- Problems in mechanics with Lie groups as configuration spaces

Example from Curve Theory

Consider a curve γ in 3 dimensions.

$$\gamma : [0, t_f] \;\rightarrow\; \mathbb{R}^3 \tag{1}$$

where $t \in [0, t_f]$ is a parametrization. A classical problem is to understand the invariants of such a curve, under arbitrary Euclidean transformations.
First, restrict to regular curves, i.e., $\dfrac{d\gamma}{dt} \neq 0$ on $[0, t_f]$. Then

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt'} \right\| dt' \tag{2}$$

is the arc-length of the segment from 0 to t.
From regularity, one can switch to a parametrization in terms of s.

Example from Curve Theory

$$\text{speed } \nu = \left\|\frac{d\gamma}{dt}\right\| = \frac{ds}{dt}$$

$$\text{Tangent vector } T = \gamma' = \frac{d\gamma}{ds} = \frac{1}{\nu}\frac{d\gamma}{dt}$$

Thus s-parametrized curve has unit speed.

If curve is C^3 and $T' = \gamma'' \neq 0$ then we can construct Frenet-Serret frame $\{T, N, B\}$ where

$$N = \frac{T'}{||T'||} \tag{3}$$

and

$$B = T \times N \tag{4}$$

By hypothesis, curvature $\kappa \stackrel{\triangle}{=} ||T'|| > 0$ and torsion is defined by

$$\tau(s) \stackrel{\triangle}{=} \frac{\gamma'(s) \cdot (\gamma''(s) \times \gamma'''(s))}{(\kappa(s))^2} \tag{5}$$

Example from Curve Theory

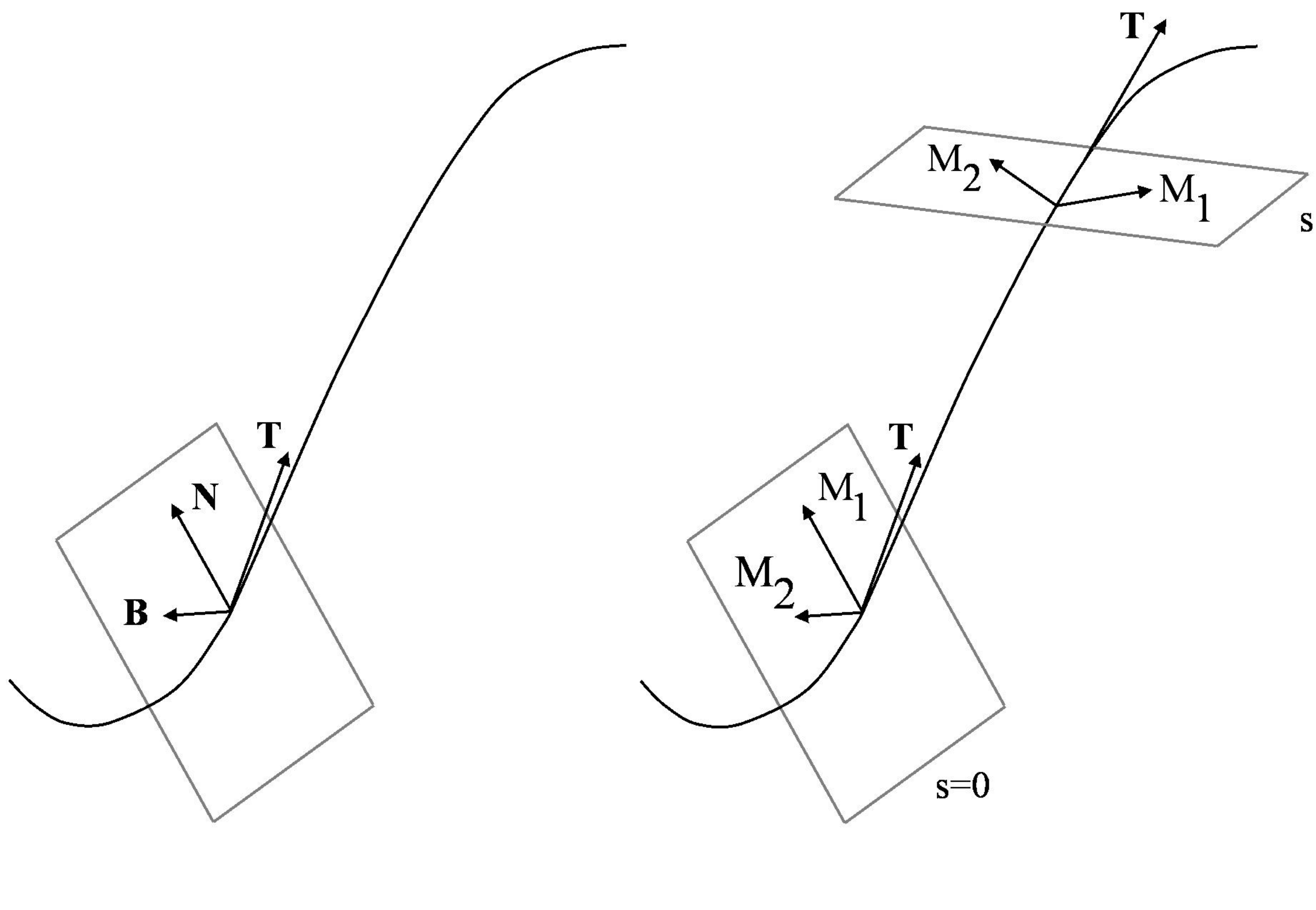
Corresponding Frenet-Serret differential equations are

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned} \tag{6}$$

Equivalently

$$\frac{d}{ds}\begin{bmatrix} T & N & B & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} T & N & B & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -\kappa & 0 & 1 \\ \kappa & 0 & -\tau & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

Example from Curve Theory



Example from Curve Theory

There is an alternative, natural way to frame a curve that only requires γ to be a C^2 curve and does not require $\gamma'' \neq 0$. This is based on the idea of a minimally rotating normal field. The corresponding frame $\{T, M_1, M_2\}$ is governed by:

$$\begin{aligned} T' &= k_1 M_1 + k_2 M_2 \\ M_1' &= -k_1 T \\ M_2' &= -k_2 T \end{aligned} \tag{8}$$

Here $k_1(s)$ and $k_2(s)$ are the natural curvature functions and can take any sign.

Frame Equations and Control

The natural frame equations can be viewed as a control system on the special Euclidean group $SE(3)$:

$$\frac{d}{ds}\begin{bmatrix} T & M_1 & M_2 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} T & M_1 & M_2 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Here, g the state in $SE(3)$ evolves under controls $k_1(s), k_2(s)$. The problem of growing a curve is simply the problem of choosing the curvature functions (controls) $k_1(\cdot), k_2(\cdot)$ over the interval $[0, L]$ where $L =$ total length. Initial conditions are needed.

Frame Equations and Control

It is possible to write everything in terms of the non-unit speed parametrization t. In that case,

$$\frac{dg}{dt} = \nu g \xi \tag{10}$$

where ν is the speed and,

$$\xi = \begin{bmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Here $\xi(\cdot)$ is a curve in the Lie algebra $se(3)$ of the group $SE(3)$, i.e., the tangent space at the identity element of $SE(3)$.
$\nu(\cdot)$, $k_1(\cdot)$ and $k_2(\cdot)$ are controls.

Examples Related to Curve Theory

- control of a unicycle can be modeled as a control problem on $SE(2)$.

- Control of a particle in $\mathbb{R}^2$ subject to gyroscopic forces can be modeled as a control problem on $SE(2)$.

Left-invariant Control Systems on Lie Groups

G is a Lie group with Lie algebra $\mathfrak{g}$.

$$L_g : G \to G \qquad R_g : G \to G$$

$$h \to gh \qquad\qquad h \to hg$$

L_g is left translation and R_g is right translation.

Control system

$$\dot{g} \;=\; T_e L_g \cdot \xi \tag{12}$$

where $\xi(\cdot)$ is a curve in the Lie algebra. For matrix Lie groups this takes the form

$$\dot{P} \;=\; PX \tag{13}$$

Left-invariant Control Systems on Lie Groups

Explicitly, let $\xi_0, \xi_1, ..., \xi_m$ be fixed elements in $\mathfrak{g}$. Consider

$$\xi(t) = \xi_0 + \sum_{i=1}^{m} u_i \cdot \xi_i \tag{14}$$

where $u_i(\cdot)$ are control inputs. Then (12) (and (13)) is manifestly a left invariant system:

$$\begin{aligned} \dot{\overbrace{L_h g}} &= T_e L_h \dot{g} \\ &= T_e L_h T_e L_g \xi \\ &= T_e L_{hg} \xi \\ &= T_e (L_h g) \xi \end{aligned}$$

Left-invariant Control systems on Lie Groups, cont'd

Input-to-state response can be written locally in t as

$$g(t) = exp\,(\psi_1(t)\tilde{\xi}_1)\,exp\,(\psi_2(t)\tilde{\xi}_2)...exp\,(\psi_n(t)\tilde{\xi}_n)g_0$$

where ψ_i are governed by Wei-Norman differential equations driven by u_i and $\{\tilde{\xi}_1, ..., \tilde{\xi}_n\}$ is a basis for $\mathfrak{g}$.

Controllability of Nonlinear Systems on Manifolds

$$\begin{aligned}
\dot{x} &= X_0(x) + \sum_{i=1}^{m} u_i X_i(x) \qquad\qquad x \in M \\
u &= (u_1, ..., u_m) \in U \subset \mathbb{R}^m \\
R^V(x_0, T) &= \{x \in M | \, \exists \text{ admissible control} \\
& \quad u : [0, T] \to U \\
& \quad s.t. \; x(t, 0, x_0, u) \in V, \;\; 0 \le t \le T \\
& \quad \text{and } x(T) = x\}
\end{aligned}$$

Controllability of Nonlinear Systems on Manifolds, cont'd

System is <u>locally accessible</u> if given any $x_0 \in M$,

$$R^V(x_0 \leq T) = \bigcup_{\tau \leq T} R^V(x_0, \tau)$$

contains a nonempty open set of M for all neighborhoods V of x_0 and all $T > 0$.

System is <u>locally strongly accessible</u> if given any $x_0 \in M$, then for any neighborhood V of x_0, $R^V(x_0, T)$ contains a nonempty open set for any $T > 0$ sufficiently small.

System is <u>controllable</u>, if given any $x_0 \in M$,

$$\bigcup_{0 \leq T < \infty} R^V(x_0 \leq T) = M,$$

i.e., for any two points x_1 and x_2 in M, there exists a finite-time T and an admissible function

$$u : [0, T] \to U, \; s.t. \; x(t, 0, x_1, u) = x_2.$$

Controllability of Nonlinear Systems on Manifolds

Let $\mathcal{L}$ = smallest Lie subalgebra of the Lie algebra of vector fields on M that contains $X_0, X_1, ..., X_m$. We call this the accessibility Lie algebra.

Let $L(x) = \{span\ X(x) | X$ vector field in $\mathcal{L}\}$.

Let $\mathcal{L}_0$ = smallest Lie subalgebra of vector fields on M that contains $X_1, ..., X_m$ and satisfies $[X_0, X] \in \mathcal{L}_0\ \forall\ X \in \mathcal{L}_0$.

Let $L_0(x) = \{span\ X(x) | X$ vector field in $\mathcal{L}_0\}$.

Controllability of Nonlinear Systems on Manifolds, cont'd

Local accessibility
$\leftrightarrow \, dim \, L(x) = n, \forall x \in M.$ $(LARC)$

Local strong accessiblity $\leftrightarrow \, dim \, L_0(x) = n,$
$\forall x \in M.$

Local accessiblity $+ \, X_0 = 0 \Rightarrow$ Controllable (Chow)

Controllability on Groups

Consider the system $\sum$ given by (12), (13).

(i) We say $\sum$ is accessible from g_0 if there exists $T > 0$ such that for each $t \in (0, T)$, the set of points reachable in time $\leq t$ has non-empty interior.

(ii) We say $\sum$ is controllable from g_0 if for each $g \in G$, there exists a $T > 0$ and a controlled trajectory γ such that $\gamma(0) = g_0$ and $\gamma(T) = g$.

(iii) We say $\sum$ is small time locally controllable $(STLC)$ from $g_0 \in G$ if there exists a $T > 0$ such that for each $t \in (0, T)$, g_0 belongs to the interior of the set of points reachable in time $\leq t$.

Controllability on Lie Groups

Let $\mathcal{U}$ = admissible controls be either

$\mathcal{U}_u$, $\mathcal{U}_\gamma$, or $\mathcal{U}_b$,

where

(i) $\mathcal{U}_u$ = class of bounded measurable functions on $[0, \infty]$ with values in $\mathbb{R}^m$.

(ii) $\mathcal{U}_\gamma$ = subset of $\mathcal{U}$ taking values in unit n-dimensional cube.

(iii) $\mathcal{U}_b$ = subset of $\mathcal{U}$ with components piecewise constant with values in $\{-1, 1\}$.

Controllability on Lie Groups, cont'd

Theorem (Jurdjevic–Sussmann, 1972)

If $\xi_0 = 0$, then controllable with $u \in \mathcal{U}$ iff $\{\xi_1, ..., \xi_m\}_{L.A.} = \mathfrak{g}$. If $\mathcal{U} = \mathcal{U}_u$ then controllable in arbitrarily short time.

Theorem (Jurdjevic-Sussman, 1972).

G compact and connected.
Controllable if $\{\xi_0, \xi_1, ..., \xi_m\}_{L.A.} = \mathfrak{g}$.
There is a bound on transfer time.
Semisimple $\Rightarrow$ tight bound.

Constructive Controllability

Underactuated systems:

$$m < dim\ (G)$$

Can we get yaw out of pitch and roll? Yes – exploit non-commutativity of $SO(3)$.

Specific idea: If drift-free, then oscillatory, small amplitude controls together with an application of averaging theory yields <u>area rule</u> and constructive techniques.

R.W. Brockett (1989), Sensors and Actuators, <u>20</u>(1-2): 91-96.

N.E. Leonard (1994), Ph.D. thesis, University of Maryland

N.E. Leonard and P.S. Krishnaprasad (1995), IEEE Trans. Aut. Contrl, 50(9): 1539-1554.

R.M. Murray and S. Sastry (1993), IEEE Trans. Aut. contrl, 38(5):700-716

Mechanical Systems on Lie Groups

– Lie groups as configuration spaces of classical mechanical systems.

– Lagrangian mechanics on TG.

– Hamiltonian mechanics on T^*G.

Key finite dimensional examples:

(a) The rigid body with Lagrangian

$$L = \frac{1}{2}\Omega \cdot \mathbb{I}\Omega - V$$

where

$$\begin{aligned} \Omega &= \text{body angular velocity} \\ &\quad \text{vector } \in \mathbb{R}^3 \\ \mathbb{I} &= \text{moment of inertia tensor} \end{aligned}$$

Mechanical Systems on Lie Groups, cont'd

(Here, body has one fixed point.)

For heavy top, potential $V = -mg \cdot R\chi$ where

$$\begin{aligned} \chi &= \text{body-fixed vector from point of} \\ &\quad \text{suspension to center of mass.} \\ g &= \text{gravity vector.} \end{aligned}$$

Mechanical Systems on Lie Groups

(b) The rigid body with Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\Omega \cdot \mathbb{I}\Omega + \Omega \cdot D\nu \\ &\quad + \frac{1}{2}\nu \cdot M\nu + mg \cdot R\eta \end{aligned}$$

where

$$\begin{aligned} \nu &= \text{rectilinear velocity} \\ \eta &= \text{body-fixed vector from} \\ &\quad\ \text{center of buoyancy to center} \\ &\quad\ \text{of gravity,} \end{aligned}$$

where body is immersed in a perfect fluid under irrotational flow. $\mathbb{I}, M$ depend also on shape of body due to 'added mass effect'.

Mechanical Systems on Lie Groups

Garret Birkhoff was perhaps the first to discuss the body-in-fluid problem as a system on the Lie group $SE(3)$. (See HASILFAS, 2nd edition, (1960), Princeton U. Press).

Controlled Mechanical Systems on Lie Groups

Hovercraft (planar rigid body with vectored thruster)

$$\begin{aligned}
\dot{P}_1 &= P_2\Pi/I + \alpha u \\
\dot{P}_2 &= -P_1\Pi/I + \beta u \\
\dot{\Pi} &= d\beta u \\
\dot{R} &= R\frac{\widehat{\Pi}}{I}; \qquad \widehat{\Pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Pi \\
\dot{r} &= R\frac{P}{m}; \qquad R = Rot(\theta)
\end{aligned}$$

Observe that the first three equations involve neither R nor r.

Controlled Mechanical Systems on Lie Groups, cont'd

This permits reduction to the first three, a consequence of $SE(2)$ symmetry of the planar rigid body Lagrangian and the "follower load" aspect of the applied thrust.

Related work by N. Leonard, students and collaborators.

Controllability of Mechanical Systems on Lie Groups

In general, on T^*G, $X_o \neq 0$. We need the idea of Poisson Stability of drift.

X is smooth complete vector field on M.
ϕ_t^X is flow of X.

$p \in M$ is positively Poisson stable for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$ such that

$$\phi_t^X(p) \in V_p.$$

X is called positively Poisson stable if the set of Poisson stable points of X is dense in M.

Controllability of Mechanical Systems on Lie Groups

A point $p \in M$ is a non-wandering point of X if for any $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$ such that

$$\phi_t^X(V_p) \cap V_p \neq \emptyset$$

X is Poisson stable $\Rightarrow$ nonwandering set $= M$.
X is $WPPS$ if nonwandering set of X is M.

Controllability of Mechanical Systems on Lie Groups

Theorem (Lian, Wang, Fu, 1994):

Control set U contains a 'rectangle'.
X_0 is $WPPS$.
Then, $LARC \Rightarrow$ Controllability

Poincare recurrence theorem

$\Rightarrow$ time independent Hamiltonian vector field on a bounded symplectic manifold is $WPPS$.

Controllability of Mechanical Systems on Lie Groups, cont'd

Theorem (Manikonda, Krishnaprasad, 2002):

Let G be a Lie group and $H : T^*G \to \mathbb{R}$
a left-invariant Hamiltonian.

(i) If G is compact, the coadjoint orbits of $\mathfrak{g}^* = T^*G/G$ are bounded and Lie-Poisson reduced dynamics $X_{\widehat{H}}$ is $WPPS$.

(ii) If G is noncompact, then the Lie-Poisson reduced dynamics is $WPPS$ if there exists a function $V : \mathfrak{g}^* \to \mathbb{R}$ such that V is bounded below and $V(\mu) \to \infty$, as $||\mu|| \to \infty$ and $\dot{V} = 0$ along trajectories of the system.

Here $\widehat{H}$ is the induced Hamiltonian on the quotient manifold $\mathfrak{g}^* = T^*G/G$.

Controllability of Mechanical Systems on Lie Groups, cont'd

Controllability of reduced, controlled dynamics on $\mathfrak{g}^*$ is of interest and can be inferred in various cases by appealing to the above theorems of (Lian, Wang and Fu, 1994) and (Manikonda and Krishnaprasad, 2002).

This leaves unanswered the question of controllability of the full dynamics on T^*G. To sort this out, we need $WPPS$ on T^*G.

Controllability of Mechanical Systems on Lie Groups

Theorem (Manikonda and Krishnaprasad, 2002):

Let G be a compact Lie group whose Poisson action on a Poisson manifold M is free and proper. A G-invariant hamiltonian vector field X_H defined on M is $WPPS$ if there exists a function $V : M/G \to \mathbb{R}$ that is proper, bounded below, and $\dot{V} = 0$ along trajectories of the projected vector field $X_{\hat{H}}$ defined on M/G.

To use this result, we see whether $LARC$ holds on M. Then appealing to theorem above, if we can conclude $WPPS$ of the drift vector field X_H then controllability on M holds.

Controllability of Mechanical Systms on Lie Groups, cont'd

Note:

For hovercraft and underwater vehicles, G is not compact, but a semidirect product. See Manikonda and Krishnaprasad, (2002), Automatica 38: 1837-1850.

Controllability of Mechanical Systems on Lie Groups

Definition:

$$\begin{aligned} H &= \text{kinetic energy.} \\ \text{Thus} \quad \dot{g} &= T_e L_g(\mathbb{I}^{-1}\mu) \\ \dot{\mu} &= \Lambda(\mu)\nabla\tilde{H} + \sum_{i=1}^{m} u_i f^i \end{aligned}$$

Here $\Lambda(\mu) =$ Poisson tensor on $\mathfrak{g}^*$.
$\tilde{H} : \mathfrak{g}^* \to \mathbb{R}$ given by

$$H^*(\mu) = \frac{1}{2}\mu \cdot \mathbb{I}^{-1}\mu$$

$$\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^* \quad \text{inertia tensor.}$$

We say that the system is equilibrium controllable if for any $(g_1, 0)$ and $(g_2, 0)$ there exists a time $T > 0$ and an admissible control

Controllability of Mechanical System on Lie Groups, cont'd

$$u : [0, t] \to U$$

such that the solution

$$(g(t), \mu(t))$$

satisfies

$$\begin{aligned} (g(0), \mu(0)) &= (g_1, 0) \quad \text{and} \\ (g(T), \mu(T)) &= (g_2, 0) \end{aligned}$$

This concept was introduced by Lewis and Murray (1996).

Controllability of Mechanical Systems on Lie Groups

Theorem (Manikonda and Krishnaprasad, 2002):

The mechanical system on T^*G with kinetic energy hamiltonian is controllable if

(i) the system is equlibrium controllable, and

(ii) the reduced dynamics on $\mathfrak{g}^*$ is controllable

Variational Problems on Lie Groups

Any smooth curve $g(\cdot)$ on a Lie group can be written as

$$\dot{g}(t) = T_e L_g \xi(t)$$

where $\xi(t) =$ curve in Lie algebra $\mathfrak{g}$ defined by

$$\xi(t) = \left(T_e L_{g(t)}\right)^{-1} \dot{g}(t)$$

Given a function l on $\mathfrak{g}$ one obtains a left invariant Lagrangian L on TG by left translation. Conversely, given a left-invariant Lagrangian on TG, there is a function $l : \mathfrak{g} \to \mathbb{R}$ obtained by restricting L to the tangent space at identity.

Variational Problems on Lie Groups, cont'd

With these meanings for ξ, L, l we state:

Theorem(Bloch, Krishnaprasad, Marsden, Ratiu, 1996):

The following are equivalent:

(i) $g(t)$ satisfies the Euler Lagrange equations for L on TG.

(ii) The variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t)) = 0$$

holds, for variations with fixed end-points.

(iii) The Euler-Poincaré equations hold:

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = ad^*_\xi \frac{\delta l}{\delta \xi}$$

(iv) The variational principle

$$\delta \int_a^b l(\xi(t))dt = 0$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta\xi = \dot{\eta} + [\xi, \eta]$$

where η vanishes at end-points.

Remark: In coordinates

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^d} = C^b_{ad}\frac{\partial l}{\partial \xi^b}\xi^a$$

where C^b_{ad} are structure constants of $\mathfrak{g}$ relative to a given basis, and ξ^a are the components of ξ relative to this basis.

Remark: Let $\mu = \dfrac{\partial l}{\partial \xi}$;
Let $h(\mu) = <\mu, \xi> - l(\xi)$ be the Legendré transform,
and assume that $\xi \to \mu$ is a diffeomorphism.

Then

$$\frac{d\mu}{dt} = ad^*_{\delta h/\delta \mu}\mu$$

the Lie-Poisson equations on $\mathfrak{g}^*$.

These are equivalent to the Euler-Poincaré equations.

Optimal Control for Left-Invariant Systems

Consider the control system

$$\dot{g} = T_e L_g \xi_u$$

where each control $u(.)$ determines a curve $\xi_{u(.)} \subset \mathfrak{g}$. Here we limit ourselves to

$$\xi_u(t) = \xi_0 + \sum_{i=1}^{m} u_i(t)\xi_i$$

where $\left\{\xi_0, \xi_1, ..., \xi_{m+1}\right\}$ spans an m-dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}$.
$m + 1 \leq n = dim\ G = dim\ \mathfrak{g}$.

Consider an optimal control problem

$$\min_{u(.)} \int_0^T L(u)dt$$

subject to the condition that $u(\cdot)$ steers the control system from g_0 at 0 to g_1 at T. In general, T may be fixed or free.

Optimal Control for Left-Invariant Systems

Here we fix T. Clearly, the Lagrangian L is G-invariant.

It is the content of the maximum principle that optimal curves in G are base integral curves of a hamiltonian vector field on T^*G. To be more precise, let $\tau_G : TG \to G$ and $\tau_g^* : T^*G \to G$ be bundle projections.

Define

$$\begin{aligned} \mathcal{H}^\lambda &= \mathcal{H}^\lambda(\alpha_g, u) \\ &= -\lambda L(u) + < \alpha_g, T_e L_g \cdot \xi_u > \end{aligned}$$

where $\lambda = 1$ or 0, and $\alpha_g \in T^*G$.

Maximum Principle:

Let u_{opt} be a minimizer of the cost functional and let $g(.)$ be the corresponding state trajectory in G. Then, $g(t) = \tau_G^*(\alpha_g(t))$ for an integral curve α_g of the hamiltonian vector field $X_{H_\lambda}^{u_{opt}}$ defined for $t \in [0, T]$ such that:

Optimal Control for Left-invariant Systems, cont'd

(a) If $\lambda = 0$ than α_g is not the zero section of T^*G on $[0, T]$.

(b) $H^\lambda(\alpha_g, u_{opt}) = \sup_{u \in U} \mathcal{H}^\lambda(\alpha_g, u)$ for t almost everywhere in $[0, T]$. Here $U =$ space of values of controls. (We consider $U = \mathbb{R}^m$ below.)

(c) If the terminal T is fixed then $H^\lambda(\alpha_g, u_{opt}) =$ constant and if T is free, then $H^\lambda(\alpha_g, u_{opt}) = 0 \quad \forall t \in [0, T]$. Trajectories corresponding to $\lambda = 0$ are called abnormal extremals and they occur but can be ruled out by suitable hypotheses. We stick to the setting of regular extremals ($\lambda = 1$).

Optimal Control for Left-invariant Systems, cont'd

We calculate the first order necessary conditions:

$$-\frac{\partial L}{\partial u_i} + \frac{\partial}{\partial u_i} < \alpha_g, T_e L_g \xi_u >= 0 \quad i = 1, 2, ..., m,$$

But

$$\begin{aligned} < \alpha_g, T_e L_g \xi_u > &= < \alpha_g, T_e L_g \left(\xi_0 + \sum_{i=1}^{m} u_i \xi_i \right) > \\ &= < T_e L_g^* \alpha_g, \, \xi_0 + \sum_{i=1}^{m} u_i \xi_i > \\ &= < \mu, \xi_0 > + \sum_{i=1}^{m} u_i < \mu, \xi_i > \end{aligned}$$

Thus

$$-\frac{\partial L}{\partial u_i} + < \mu, \xi_i >= 0 \quad i = 1, 2, ..., m$$

Optimal Control for Left-invariant Systems, cont'd

At this stage, the idea is to solve for u_i and plug into

$$\mathcal{H}^\lambda = -L(u) + < \mu, \xi_0 > + \sum_{i=1}^{m} u_i < \mu, \xi_i >$$

to get a G-invariant hamiltonian which descends to a hamiltonian h on $\mathfrak{g}^*$.

Optimal Control for Left-invariant Systems, cont'd

In the special case

$$L(u) = \frac{1}{2}\sum_{i=1}^{m} I_i u_i^2,$$

we get $u_i = \dfrac{< \mu, \xi_i >}{I_i}$

and $\quad h = < \mu, \xi_0 > + \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} \dfrac{< \mu, \xi_i >^2}{I_i}.$

One solves the Lie-Poisson equation

$$\frac{d\mu}{dt} = ad^*_{\delta h/\delta \mu}\mu$$

to obtain μ as a function of t.
Then substitute back into

$$u_i = \frac{< \mu, \xi_i >}{I_i}$$

to get controls that satisfy first order necessary conditions.

Optimal Control for Left-invariant Systems, cont'd

Integrable Example (unicycle) on $SE(2)$

$$\begin{aligned} \dot{g} &= g\,(\xi_1 u_1 + \xi_2 u_2) \\ \text{where } \xi_1 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \xi_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ L(u) &= \frac{1}{2}(u_1^2 + u_2^2) \end{aligned}$$

Then

$$
\begin{aligned}
h(\mu) &= \frac{1}{2}(\mu_1^2 + \mu_2^2) \\
\dot{\mu}_1 &= -\mu_2 \mu_3 \\
\dot{\mu}_2 &= \mu_1 \mu_3 \\
\dot{\mu}_3 &= -\mu_1 \mu_2
\end{aligned}
$$

Invariants

$$
\begin{aligned}
c &= \mu_2^2 + \mu_3^2 \quad \text{(casimir)} \\
h &= (\mu_1^2 + \mu_2^2)/2 \quad \text{(hamiltonian)}
\end{aligned}
$$

Optimal Control for Left-invariant Systems, cont'd

Then

$$\ddot{\mu}_2 \;+\; (2h+c)\mu_2 - 2\mu_2^3 = 0$$

$$\text{(anharmonic oscillator)}$$

$$\mu_2(t) \;=\; \beta Sn(\gamma(t-t_0),k)$$

where $Sn(u,k)$ is Jacobi's elliptic sine function, γ $s.t.$

$$\gamma^2 < (2h+c) < 2\gamma^2$$

t_0 is arbitrary and

$$k^2 \;=\; \frac{2h+c}{\gamma^2} - 1$$

$$\beta^2 \;=\; 2h+c-\gamma^2$$

Then

$$
\begin{aligned}
\mu_1 &= \sqrt{2h - \mu_2^2} \\
\mu_3 &= \sqrt{c - \mu_2^2} \\
u_1 &= \mu_1 \\
\text{and } u_2 &= \mu_2.
\end{aligned}
$$

Optimal Control for Left-invariant Systems

In the above example one can show that there are no abnormal extremals.

This example is prototypical of a collection of integrable cases. Where integrability does not hold, one can still investigate the Lie-Poisson equations numerically.