ENEE 661
Nonlinear Control Systems
Lecture 2

PSK
02.09.00.

Planar Systems

The study of planar systems is best explored by the study of planar linear systems. Consider the system in $\mathbb{R}^2$

$$\dot{x} = Ax,$$

where A is a 2×2 real constant matrix. From linear algebra, there is a real nonsingular matrix P such that

$$J = PAP^{-1}$$

is one of the following three forms:

(i) $$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(ii) $$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

(iii) $$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

These are the possible real Jordan forms.

Prepared by PSK

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The change of variables $y = Px$ defines the system

$$\dot{y} = Jy$$

with the solutions related by the formulas,

$$x(t) = P^{-1} y(t)$$

$$= P^{-1} e^{Jt} y(0)$$

$$= P^{-1} e^{Jt} P\, x(0).$$

Everything about the behavior of $x(\cdot)$ can be determined from that of $y(\cdot)$

In case $\underline{(i)}$

$$y_i(t) = e^{\lambda_i t} y_i(0) \quad i = 1, 2.$$

In case $\underline{(ii)}$, $y_1(t) = e^{\lambda t}(y_1(0) + t\, y_2(0))$

$$y_2(t) = e^{\lambda t} y_2(0)$$

$$\Rightarrow y_1 = \frac{y_1(0)}{y_2(0)} y_2 + \frac{1}{\lambda} \ln\left(\frac{y_2}{y_2(0)}\right) \cdot y_2$$

In case $\underline{(iii)}$, letting $r = \sqrt{y_1^2 + y_2^2}$ and $\phi = \arctan(y_2/y_1)$, we get

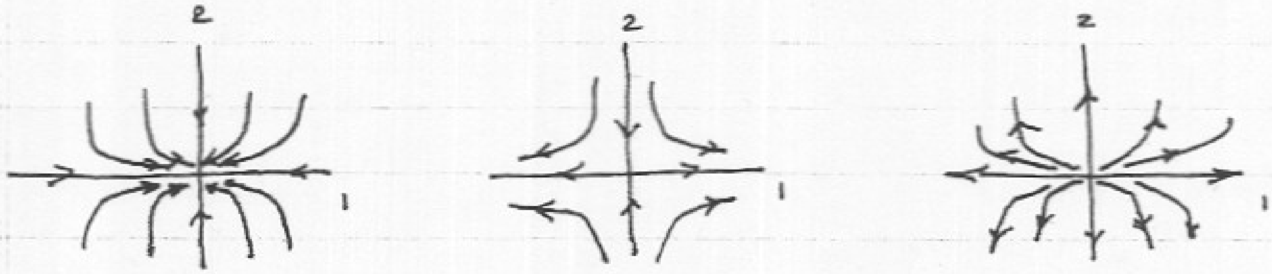
$$\dot{r} = \alpha r$$

$$\dot{\phi} = -\beta$$

Thus in this last case r spirals in or out of 0 according as $\alpha > 0$ or $\alpha < 0$.

The behavior of the linear ~~equation~~ system around the origin is thus captured by the classification in ~~fi~~ Figure 1, upto a nonsingular linear transformation P. ~~Sta~~ In the figure we represent behavior of (y_1, y_2).

(i)

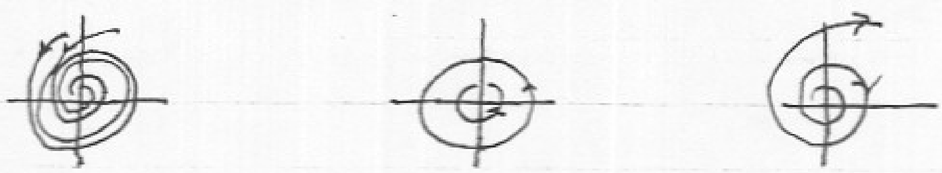


$\lambda_2 < \lambda_1 < 0$ stable node | $\lambda_2 < 0 < \lambda_1$ saddle | $\lambda_2 > \lambda_1 > 0$ unstable node

(ii)

(stable) $\lambda < 0$ improper node $\lambda > 0$ (unstable)

(iii)



stable focus $(\alpha < 0)$ center $(\alpha = 0)$ unstable focus $(\alpha > 0)$

<u>Figure 1</u> Classification Planar/Linear

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It is clear that small perturbations of A would alter the phase portraits of the center and the improper node to ~~to~~ one of the remaining 5 possibilities. These latter are the generic phase portraits near 0. The generic picture in the linear case carries over to the nonlinear setting.

Consider the nonlinear <u>planar</u> system,

$$\dot{x} = f(x) .$$

Denote a solution starting at x by $\phi_t^f(x)$. The superscript here keeps track of the system in question. Thus,

$$\frac{d}{dt}\left(\phi_t^f(x)\right) = f\left(\phi_t^f(x)\right)$$

$$\phi_0^f(x) = x . \qquad \text{(initial condition)}$$

Let x_e be an equilibrium point, i.e

$$f(x_e) = 0 .$$

Thus $\phi_t^f(x_e) = x_e \quad \forall \; t \in \mathbb{R}$.

Let us denote the linearization of f at x_e by A;

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x = x_e}$$

The solution to the linearization ~~is given by~~

$$\dot{x} = Ax$$

is given by

$$\phi_t^A(x) = e^{tA}x .$$

The map $\phi_t^f : \mathbb{R}^2 \to \mathbb{R}^2$ is called the flow map and $\{\phi_t^f : t \in \mathbb{R}\}$ the flow of the system

$$\dot{x} = f(x).$$

If x_e is an equilibrium such that A has no eigenvalues on the imaginary axis, then we call x_e an hyperbolic equilibrium point. We now have the connection between the linearization and the nonlinear system.

Theorem (Hartman – Grobman)

Consider the nonlinear system

$$\dot{x} = f(x) \qquad , \qquad x \in \mathbb{R}^n$$

with hyperbolic equilibrium x_e. Let $A = \left(\frac{\partial f}{\partial x}\right)_{x_e}$ denote the linearization of f.

Let $\{\phi_t^f\}$ denote the flow of the nonlinear system. Then, there exists a map

$$F : B_\delta \subset \mathbb{R}^n \to \mathbb{R}^n .$$

where $B_\delta = \{x : \|x - x_e\| < \delta\}$ is a ball defining a sufficiently small neighborhood of x_e, such that $F(x_e) = 0$, F is one-to-one, and onto $F(B_\delta)$ and the map F as well as its inverse F^{-1} are <u>continuous</u>, (We call F a <u>homeomorphism</u>.) such that,

$$F(\phi_t^f(x)) = e^{tA} F(x) \qquad x \in B_\delta$$

or more ~~succintly~~ succinctly

$$\boxed{\phi_t^f = F^{-1} \circ e^{tA} \circ F}$$

▨

<u>Remark:</u> We say that the flow ϕ_t^f is <u>conjugate</u> to that of the flow e^{tA}. We have tacitly assumed that the nonlinear system has a well-defined solution for all time. This is not necessary for the statement of the theorem. We only need existence for $|t| < T$, some $T > 0$.

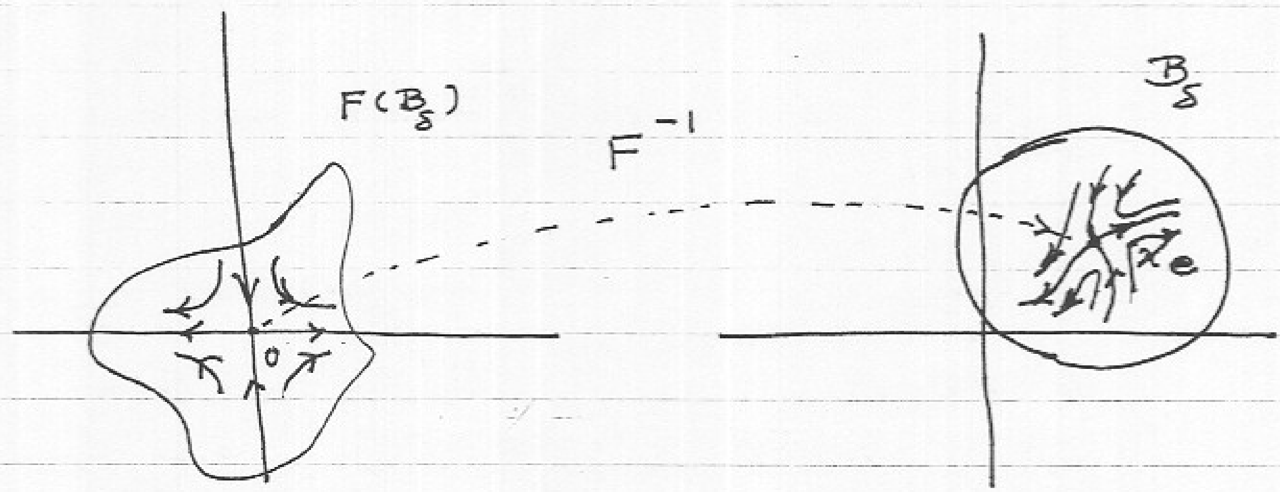

Figure 2

Hartman–Grobman Theorem.

The phase portrait of the nonlinear system near an equilibrium x_e is a <u>distorted</u> (by F^{-1}) <u>version</u> of the phase portrait of the linearization near 0.

<u>Remark</u>: The hyperbolicity assumption of the Hartman–Grobman Theorem is important; as the following illustrates.

<u>Example</u> Consider the system

$$\dot{x}_1 = -x_2 - \mu x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2 (x_1^2 + x_2^2).$$

The linearization at $(0,0)$ is just the harmonic oscillator with center

(0,0). But, from the polar coordinate representation of the system,

$$\dot{r} = -\mu r^3$$
$$\dot{\theta} = 1 \quad ,$$

So the solutions to the nonlinear system ~~spiral~~ in (out) towards (away from) zero for $\mu > 0$ ($\mu < 0$).

Closed Orbits of a Dynamical System

Definition : Consider the nonlinear system with flow ϕ_t^f. We say that

x is a nontrivial periodic point of period T if $\phi_T^f(x) = x$ for some $T > 0$

and T is the smallest such time. The trajectory γ through a periodic point is called a periodic orbit;

$$\gamma = \{ \phi_t^f(x) : t \geq 0 \} .$$

Such orbits are closed (as curves).

Finding periodic orbits is hard. But in $\mathbb{R}^2$ we have a sufficient condition

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<u>Definition</u>: A region $M \subset \mathbb{R}^n$ is positively (negatively) invariant for the flow ϕ_t^f, if for each $x \in M$,

$$\phi_t^f(x) \in M$$

for all $t \geq 0$ $(t \leq 0)$.

<u>Theorem</u>: (Poincaré-Bendixson)

Consider the continuous time dynamical system <u>in the plane</u>

$$\dot{x} = f(x).$$

Let M be a closed and bounded positively invariant set for the flow $\{\phi_t^f : t \geq 0\}$. Suppose M ~~contain~~ <u>does not</u> contain any equilibria of the given system. Then M contains a <u>closed orbit</u>.

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The following example of a fundamental biochemical process called glycolysis, taken from S. H. Strogatz (Nonlinear Dynamics and Chaos) is a nice illustration of the Poincaré–Bendixson Theorem.

Glycolysis in living cells generates energy through the breaking down of sugar. In intact yeast cells as well as in yeast or muscle extracts, glycolysis proceeds, under suitable conditions, in an oscillatory fashion, with the periodic rise and fall of the concentrations of various intermediates. A simple ~~kinematic~~ kinetic model due to Sel'kov (1968) (Eur. J. Biochem. 4:79), in dimensionless form is given by

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

where x and y are, respectively, the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6-phosphate), $a, b > 0$ are kinetic parameters.

To show that this system has a periodic solution via the Poincaré–Bendixson Theorem, one needs to find a positively invariant set for the system, that contains no equilibria, that is closed and bounded. Such a set would also be called a <u>trapping region</u> for ~~th~~ the system. There will be conditions on a and b as a result. We construct a trapping region in Figure 1.

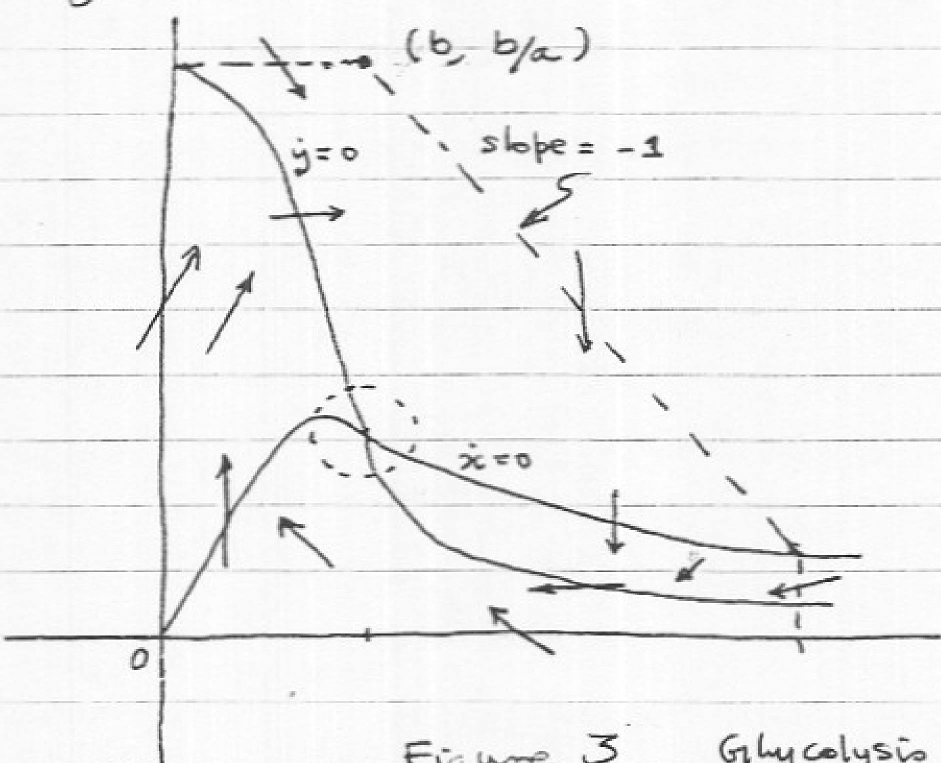

Figure 3 Glycolysis Model

In the figure, the solid curves $\dot{x}=0$ (equivalently $y = x/(a+x^2)$), and $\dot{y}=0$ (equivalently $y = b/(a+x^2)$), are the <u>null-clines</u>. On these curves the direction fields are marked by vertical and horizontal arrows respectively. Ignore the dotted circle for the moment. The intersection of the null-clines gives the

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unique equilibrium point $\left(b, \frac{b}{a+b^2}\right)$.
The five-sided figure bounded by the horizontal and vertical axes and the three dotted straight line segments is a trapping region for the system in the sense that, once a trajectory enters this region, it never leaves it. To see this, convince yourself that the arrows are drawn correctly on the boundaries of the region.
Hint: Verify that in the region <u>above</u> the <u>null cline $\dot{x} = 0$</u>, we obtain $\dot{x} > 0$ and <u>below</u> it we obtain $\dot{x} < 0$; in the region to the left of the null cline $\dot{y} = 0$ but to the right of $x = 0$, $\dot{y} > 0$, and to the right of the nullcline $\dot{y} = 0$ and above $y = 0$ we obtain $\dot{y} < 0$. On the diagonal dotted line of slope -1,

$$\begin{aligned} \dot{x} - (-\dot{y}) &= -x + ay + x^2 y \\ &\quad + b - ay - x^2 y \\ &= b - x, \end{aligned}$$

implying $-\dot{y} > \dot{x}$ if $x > b$ which is the case.

~~The notion that~~ We cannot conclude from this that there is a periodic solution in the trapping region. This is because we have an equilibrium point in this region — violating one of the hypotheses of the Poincaré Bendixson Theorem. What do we do?

Well, if the equilibrium $\left(b, \frac{b}{a+b^2}\right)$ is an <u>unstable</u> node or a focus, then on a small dotted circle surrounding the equilibrium, all arrows will be pointing outward. Then one can conclude that the trapping region <u>minus</u> the <u>open</u> disk ~~bounded~~ by the dotted circle, is a closed and bounded positively invariant set (i.e. a smaller trapping region) containing no equilibria. Hence it must contain a periodic orbit by PBT.

So what are the conditions for the equilibrium to be an unstable node or focus?

Linearize the dynamics at the equilibrium to get

$$A = (Df)_e = \left. \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a+x^2) \end{pmatrix} \right|_{\substack{x = b \\ y = b/(a+b^2)}}$$

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$\det(A) = a + b^2$ and $tr(A)$ = trace of A $= -\dfrac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}$.

Thus the equilibrium is unstable if $tr(A) > 0$ and stable if $tr(A) < 0$. The stability regions in parameter space are separated by the curve

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right)$$

See Figure

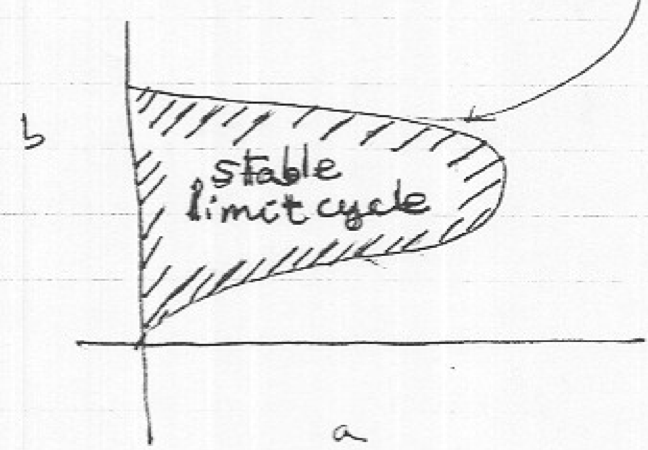

Figure 4.
Stability region in parameter space

Recall the classification diagram in parameter space for planar linear systems ~~in~~ in Figure . For a 2×2 matrix $A = [a_{ij}]$ the characteristic polynomial

$$\chi_A(s) = \det \begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix}$$

$$= (s - a_{11})(s - a_{22}) - a_{12}a_{21}$$

$$= s^2 - s(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21})$$

$$= s^2 - \tau s + \Delta$$

where $\tau = tr(A)$ and $\Delta = \det(A)$. The discriminant for the characteristic equation is $\tau^2 - 4\Delta$

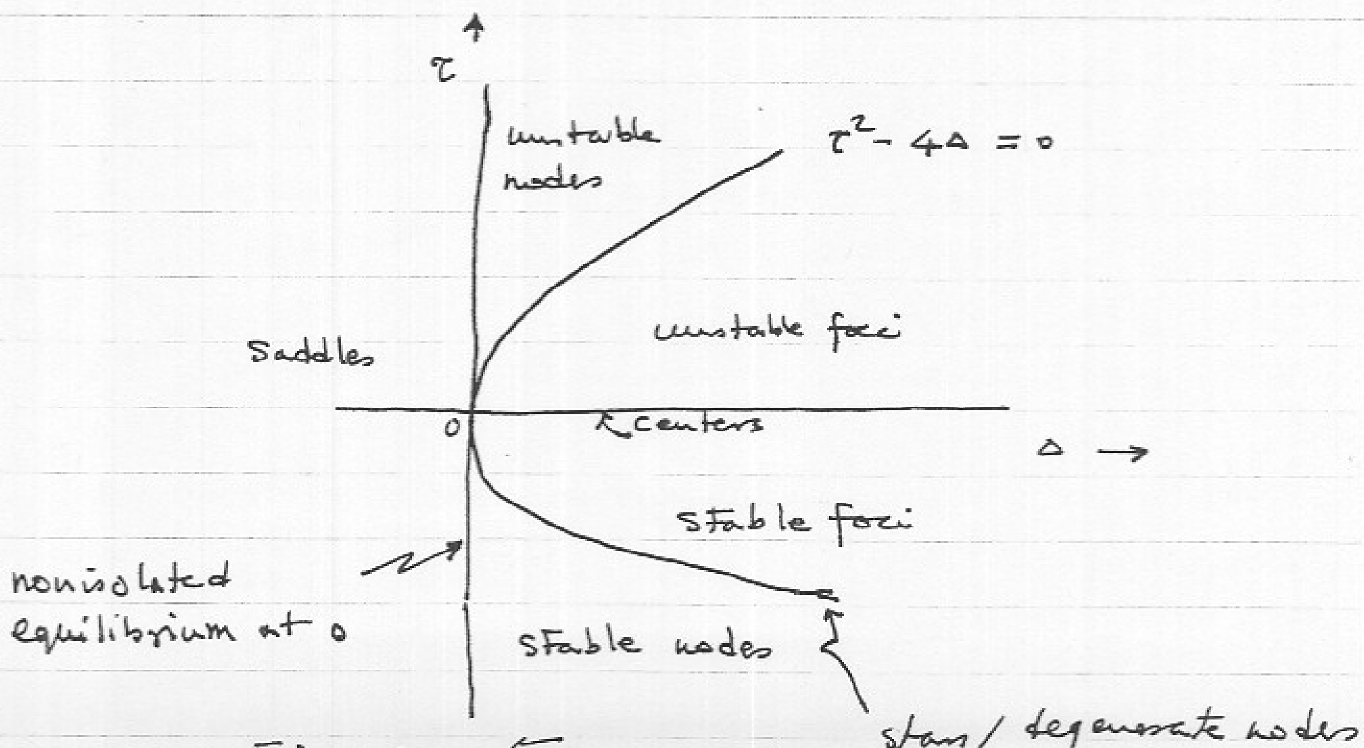

Figure 5