

For a nonlinear system $\dot{x} = f(x)$, with equilibrium x_e , we define the region of attraction of x_e denoted $R_A(x_e)$ to be the set

$$R_A(x_e) = \left\{ x_0 : \phi(t, x_0) = \text{sol}^n \text{ to } \dot{x} = f(x) \right. \\ \left. \rightarrow x_e \text{ as } t \rightarrow \infty \right\}$$

If x_e is an asymptotically stability equilibrium, then one can prove that $R_A(x_e)$ is an open, invariant set containing x_e . When is it the whole state space? Clearly, if $R_A(x_e) = \mathbb{R}^n$, then x_e is the unique equilibrium. Additionally, one can rule out all other invariant sets (e.g. periodic orbits). The following is a set of sufficient conditions.

Theorem (Barabashin - Krasovskii)

Let $x=0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function such that

- (i) $V(0) = 0$, $V(x) > 0$ $x \neq 0$.
- (ii) $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ (radial unboundedness)
- (iii) $\dot{V}(x) < 0$ $\forall x \neq 0$

Then $x=0$ is globally asymptotically stable ($\Rightarrow R_A(0) = \mathbb{R}^n$)

Proof: Let $p \in \mathbb{R}^n$, $V(p) = c > 0$.

From (ii) there exists $r > 0$ s.t.

See pages 10 & 11 of these notes

No loss of generality in picking $x_e = 0$; null change of coordinates trick.

$$\{x \in \mathbb{R}^n \mid V(x) \leq c\} = \Omega_c \subset B_r.$$

Thus Ω_c is bounded, (and closed). By LaSalle's invariance principle $x(t)$, a trajectory beginning in Ω_c , $\rightarrow M =$ largest invariant set $\subset E = \{x \mid \dot{V} = 0\}$, as $t \rightarrow \infty$.

By (iii) $M = \{0\}$.

Thus 0 is asymptotically ~~stability~~ stable and globally attractive. \square

The idea of using "energy-like" functions can be used also to prove instability theorems. A fundamental result is due to Nikolai Guryevich Chetaev (1902-1959), a Soviet mechanician, who held the chair of Theoretical Mechanics at Moscow State University.

Theorem (Chetaev)

Let $x=0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V: D \rightarrow \mathbb{R}$ be a C^1 function defined on an open, connected subset D of \mathbb{R}^n which contains 0, and such that \circ

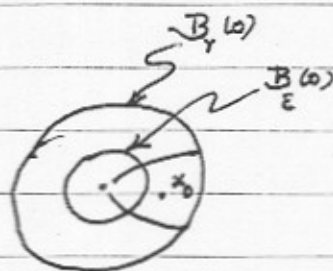
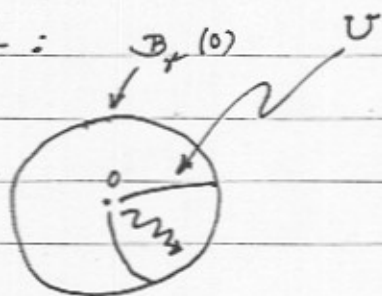
- (i) $V(0) = 0$ and for every $\varepsilon > 0$, there exists $x_0 \in B_\varepsilon(0)$ such that $V(x_0) > 0$.
- (ii) Let $U = \{x \in B_r(0) \mid V(x) > 0\}$ for $B_r \subset D$ and $\dot{V}(x) \geq \phi(\|x\|) > 0$ on U

$\phi: [0, \infty) \rightarrow [0, \infty)$ is continuous, strictly increasing with $\phi(0) = 0$.

Then ~~the~~ $x=0$ is an unstable equilibrium

(remark: such a function ϕ is called a class \mathcal{K} function)

Proof:



Let $x_0 \in \text{int}(U)$ and $V(x_0) = a > 0$.

Suppose $\|x_0\| = r_0$. Consider the annulus

$$\{x: \varepsilon \leq \|x\| \leq r\} \quad \text{where } \varepsilon > 0$$

is such that $V(x) \leq \frac{a}{2}$ for $\|x\| \leq \varepsilon$.

Such an ε exists by continuity of V

at the origin. Then, since $\dot{V}(x(t)) > 0$

and $V(x_0) = a$, it follows that a trajectory must not enter B_ε .

The annulus $\{x: \varepsilon \leq \|x\| \leq r\}$ is a closed and bounded set and $\dot{V}(x) > 0$ in U . Since $\dot{V}(x) \geq \phi(\|x\|)$ a class \mathcal{K} function $\dot{V} \geq \delta > 0$ for some δ . We can take

$$\delta = \inf \{ \phi(\lambda) \mid \lambda \in [\varepsilon, r] \}.$$

Then,

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(s)) ds \\ &\geq a + \int_0^t \gamma ds \\ &= a + \gamma t. \end{aligned}$$

But $V(\cdot)$ is bounded in U . Thus $x(t)$ cannot remain forever in U .

In fact it leaves U at the latest by

$$T = \frac{1}{\gamma} (\sup_U V - a).$$

Now $x(t)$ cannot leave U through the surface $\{x: V(x) = 0\}$ since $V(x(t)) \geq a > 0$. Hence it must leave through the sphere $\|x\| = r$. Since this can happen for arbitrarily small $\|x_0\|$, the origin is unstable. \square

<We call such a V a Cetaev function>

Example

Consider the planar system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $|g_i(x)| \leq k \|x\|_2^2$ in a nbhd D of 0, ($\Rightarrow g_i(0) = 0$ and thus 0 is an equilibrium point). Let $V(x) = \frac{1}{2} (x_1^2 - x_2^2)$.

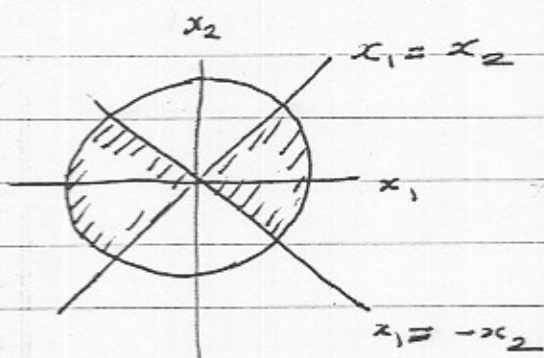
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On the line $x_2 = 0$, i.e. the x_1 axis, $V(x) > 0$ at points arbitrarily close to the origin.

Pick

$$U = \{ (x_1, x_2) \in B_r(0) \mid V(x_1, x_2) > 0 \}$$

i.e. the shaded area in the adjoining figure.



$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

$$\text{But } |x_1 g_1(x) - x_2 g_2(x)| \leq \sum_{i=1}^2 |x_i| g_i(x)$$

$$\leq 2k \|x\|_2^3 \quad (\text{by hypothesis})$$

$$\text{Hence } \dot{V}(x) \geq x_1^2 + x_2^2 - |x_1 g_1(x) - x_2 g_2(x)|$$

$$\geq x_1^2 + x_2^2 - 2k \|x\|_2^3$$

$$= \|x\|_2^2 (1 - 2k \|x\|_2)$$

Choose $r > 0$ such that $B_r(0) \subset D$ and $r < \frac{1}{2k}$. Then $\dot{V}(x) > 0$ on U .

By Chetaev's theorem the origin is unstable. (Question: what is ϕ here?) \square

Example

$$M\ddot{x} + (S + \varepsilon R)\dot{x} + Kx = 0$$

$$M = M^T > 0, \quad S = -S^T, \quad R = R^T > 0, \quad K = K^T \text{ indefinite.}$$

$$\text{Let } p = M\dot{x}$$

$$\dot{x} = M^{-1}p$$

$$\dot{p} = -(S + \varepsilon R)M^{-1}p - Kx$$

 $\varepsilon > 0.$

$$H = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} x^T K x$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p}$$

$$= Kx \cdot M^{-1}p + M^{-1}p \cdot ((-S - \varepsilon R)M^{-1}p - Kx)$$

$$= -\varepsilon p \cdot M^{-1} R M^{-1} p \quad (\because S = -S^T)$$

Pick the Cetaev function to ~~be~~ be
 $V = -H$. Convince yourself, by argument
 as in the previous example (pick U
 properly), that $(0, 0)$ is unstable in $\mathbb{R}^n \times \mathbb{R}^n$.

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There are instability results due to Lyapunov that precede the work of Cetaev and were motivating influences on Cetaev's work.

Theorem (Lyapunov - instability I)

If V is C^1 in a domain $D \subset \mathbb{R}^n$ with $0 \in D$, $f(0) = 0$ and $V(0) = 0$. Also $\dot{V}(x) > 0$ $x \neq 0$ in D , and $V(x)$ assumes positive values arbitrarily near 0, then 0 is an unstable equilibrium.

Proof: Without loss of generality, let D be a bounded domain. V is bounded in D . Let $r \leq R$, $B_r(0) \subset D$. Then $\exists x_0 \in B_r(0)$ such that $V(x_0) > 0$.

Since $\dot{V} > 0$, $V(x(t))$ can only increase and so $x(t) \not\rightarrow 0$. In fact, since $\dot{V} \geq m > 0$, $x(t) \not\rightarrow$ fixed point in $B_r(0)$. $V(x(t))$ must increase indefinitely and hence $x(t)$ must eventually leave $B_r(0)$ via some point in the boundary $\partial B_r(0)$. \square

Theorem (Lyapunov - Instability II)

Same hypothesis on V as in (I) above, and $\dot{V} = \lambda V + V^*$ where $V^* \geq 0$ in D and $\lambda > 0$. Then 0 is an unstable equilibrium.

Proof: Let $x_0 \in B_r(0)$ as in previous. $V(x_0) > 0$.

$$x(t) \equiv \phi(t, x_0)$$

$$\dot{V} = \lambda V + V^* \Rightarrow$$

$$\frac{d}{dt} (e^{-\lambda t} V) = e^{-\lambda t} V^* \geq 0$$

Hence along $x(t)$, $V \geq e^{\lambda t} V(x_0)$.

V increases indefinitely along $x(t)$

\Rightarrow instability as in previous theorem \square

Some special Lyapunov functions

Let $V = V_p(x) + V_{p+1}(x) + \dots$

be defined in a nbhd of 0, where V_k is a homogeneous polynomial of degree k .

Then the sign of V in a suitable nbhd Ω of the origin is the same as the sign of V_p .

Lemma: If p is odd V cannot be a Lyapunov function.

Proof: $x_1 = x_n u_1$; $x_2 = x_n u_2$; \dots ; $x_{n-1} = x_n u_{n-1}$

$\Rightarrow V_p = x_n^p V_p(u_1, u_2, \dots, u_{n-1}, 1)$. Keeping the u_i fixed, the sign of V_p will be the sign of x_n^p or $-x_n^p$ (one of the two but not both).

Since p is odd x_n^p or $-x_n^p$ may assume both positive and negative values near 0. So

V is not positive definite. We are tacitly assuming that we have chosen u_i s.t. $V_p(u_1, u_2, \dots, u_{n-1}, 1) \neq 0$. This is always possible since $V_p \neq 0$. \square

Note: Such power series expansions as in V above may not be defined, but we might still have a positive definite C^1 Lyapunov function e.g.

$$V(x) = \begin{cases} x^2 & x \geq 0 \\ x^4 & x \leq 0 \end{cases} .$$

Lemma (Invariant Sets)

If $x=0$ is an asymptotically stable equilibrium point, then its region of attraction $R_A(0)$ is an open, invariant set. Moreover $\partial R_A(0)$ the boundary of $R_A(0)$, is formed by trajectories.

Proof: Let $\phi(s, x)$ denote the solution to $\dot{x} = f(x)$, with initial condition $y(0) = x$, $s \in \mathbb{R}$. We wish to show that $\phi(s, x) \in R_A(0)$ whenever $x \in R_A(0)$, $\forall s \in \mathbb{R}$.

By the semigroup property of solutions to o.d.e's

$$\begin{aligned}\phi(t, x(s)) &= \phi(t, \phi(s, x)) \\ &= \phi(t+s, x)\end{aligned}$$

$$\begin{aligned}\lim_{t \rightarrow \infty} \phi(t, x(0)) &= \lim_{t \rightarrow \infty} \phi(t+s, x) \\ &\in \mathcal{R}_A(0) \quad \text{by definition.}\end{aligned}$$

Thus $\phi(s, x) \in \mathcal{R}_A(0) \forall s \in \mathbb{R}$, whenever $x \in \mathcal{R}_A(0)$. (INVARIANCE)

To prove openness of $\mathcal{R}_A(0)$, let $p \in \mathcal{R}_A(0)$. Let $T > 0$ be sufficiently large that

$$\|\phi(T, p)\| < \frac{a}{2}$$

where $a > 0$ is such that $\{x: \|x\| < a\} \subset \mathcal{R}_A(0)$

~~is not empty~~ We can choose b small enough such that $\forall q \in \{x: \|x-p\| < b\}$, the solution $\|\phi(T, p) - \phi(T, q)\| < \frac{a}{2}$.

$$\begin{aligned}\text{Hence } \|\phi(T, p)\| &\leq \|\phi(T, p) - \phi(T, q)\| + \|\phi(T, q)\| \\ &< a.\end{aligned}$$

$$\Rightarrow \phi(T, q) \in \mathcal{R}_A(0)$$

$$\Rightarrow \lim_{t \rightarrow \infty} \phi(t, q) = 0 \Rightarrow q \in \mathcal{R}_A(0).$$

For any open, invariant set M
 $x \in \partial M \Rightarrow \{x_n\} \subset M$ s.t. $\lim_{n \rightarrow \infty} x_n = x$.

Hence $\{\phi(t, x_n) : t \in \mathbb{R}\} \subset M$.

$$\lim_{t \rightarrow \infty} \phi(t, x_n) = \phi(t, x)$$

$\Rightarrow \phi(t, x)$ is an accumulation point of M ; $\forall t \in \mathbb{R}$.

But $\phi(t, x) \notin M$, since $x \in \partial M$ & M is open. Therefore $\phi(t, x) \in \partial M$ $\forall t \in \mathbb{R}$. Thus ∂M is made up of trajectories.