The flow of a vector field f satisfies $(\Phi_f^t)^{-1} = \Phi_f^{-t} = \Phi_{-f}^t$ (reversing the arrow is same as reversing flow of time). For linear vector fields $f(x) = Ax$, and $g(x) = Bx$, using

$$\Phi_f^t(x) = e^{tA}x \quad ; \quad \Phi_g^t(x) = e^{tB}x,$$

we have shown that

$$\Phi_g^{-\varepsilon}(\Phi_f^{-\varepsilon}(\Phi_g^{\varepsilon}(\Phi_f^{\varepsilon}(x_0))))$$

$$= e^{-\varepsilon B} e^{-\varepsilon A} e^{\varepsilon B} e^{\varepsilon A} x_0$$

$$= x_0 + \varepsilon^2 (BA - AB)x_0 + o(\varepsilon^2)$$

$$= x_0 + \varepsilon^2 [f, g](x_0) + o(\varepsilon^2)$$

In fact the last line holds for general <u>nonlinear</u> vector fields. The proof of this relies on an expression for the flow from using the <u>fundamental theorem</u> of integral calculus.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable. Let $g(t) = F(x + th)$. Then $g(1) = g(0) + \int_0^1 \frac{dg}{dt}\, dt$

$$= F(x) + \int_0^1 \frac{d}{dt} F(x+th)\, dt$$

$$= F(x) + \int_0^1 DF(x+th)\, h\, dt$$

(chain rule)

where $DF(y)$ denotes the linear operator defined by $DF(y)\eta = \frac{d}{d\varepsilon} F(y+\varepsilon\eta)\Big|_{\varepsilon=0}$.

(It is simply given by the Jacobian matrix

$$DF(x) = \left[\frac{\partial F^i}{\partial x_j}\right]\Big|_x \quad)$$

Now this process can be repeated as follows:
Let $g(s) = DF(x+tsh)h$. Then

$$g(1) = DF(x+th)h$$

$$= g(0) + \int_0^1 \frac{d}{ds} g(s)\, ds$$

$$= DF(x)h + \int_0^1 \frac{d}{ds} DF(x+tsh)\, h\, ds$$

$$= DF(x)h + \cancel{D^2F(h,h)}$$

$$+ \int_0^1 D^2F(x+tsh)(h,h)\, t\, ds$$

(chain rule)

where $D^2F(x+tsh)(h,h)$ is a column vector with i^{th} element $= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F^i}{\partial x_k \partial x_j}\Big|_{x+tsh} h_j h_k$

Hence. t

$$F(x+h) = F(x) + \int_0^1 \Big(DF(x)\cdot h + \int_0^1 D^2F(x+tsh)(h,h)\,ds \Big)\,dt$$

t $\qquad t$

$$= F(x) + t\,DF(x)\,h + \int_0^1 \int_0^1 t\,D^2F(x+tsh)(h,h)\,ds\,dt$$

t

$$= F(x) + t\,DF(x)\,h + \frac{t^2}{2!} D^2F(x)(h,h) + \overset{= O(t^3)}{o(t^2)}$$

(by repeating the same use of ~~(by another repetition)~~ the fundamental theorem).

Applying this process to the flow Φ_f^t and using

$$\frac{d}{dt}\Phi_f^t(x) = f\left(\Phi_f^t(x)\right)$$

or equivalently,

$$\Phi_f^t(x) = x + \int_0^t f\left(\Phi_f^\sigma(x)\right)d\sigma$$

$f(x)$

we obtain: $$\boxed{\Phi_f^t(x) = x + t f(x) + \frac{t^2}{2!} Df(x)\cancel{x} + O(t^3)}$$

<u>proof</u>:

(a) Let $g(\lambda) = f(\Phi_f^{\lambda\sigma}(x))$

then $g(1) = f(\Phi_f^{\sigma}(x))$

$g(0) = f(\Phi_f^{0}(x))$

$= f(x)$

$$f(\Phi_f^{\sigma}(x)) = g(1) = g(0) + \int_0^1 \frac{dg}{d\lambda}\, d\lambda$$

$$= f(x) + \int_0^1 \frac{d}{d\lambda} f(\Phi_f^{\lambda\sigma}(x))\, d\lambda$$

$$= f(x) + \int_0^1 \sigma Df(\Phi_f^{\lambda\sigma}(x)) f(\Phi_f^{\lambda\sigma}(x))\, d\lambda$$

$$\Rightarrow \Phi_f^{t}(x) = x + \int_0^t f(\Phi_f^{\sigma}(x))\, d\sigma$$

$$= x + \int_0^t f(x)\, d\sigma$$

$$+ \int_0^t \sigma \int_0^1 Df(\Phi_f^{\lambda\sigma}(x)) f(\Phi_f^{\lambda\sigma}(x))\, d\lambda\, d\sigma$$

(b) Let $g(\mu) = \int_0^1 Df(\Phi_f^{\lambda\mu\sigma}(x)) f(\Phi_f^{\lambda\mu\sigma}(x))\, d\lambda$

$$g(1) = g(0) + \int_0^1 \frac{dg}{d\mu}\, d\mu$$

$$= Df(x) f(x) \cdot 1 + \int_0^1 \frac{dg}{d\mu}\, d\mu$$

Hence,

$$\Phi_f^t(x) = x + t \cdot f(x) + \frac{t^2}{2!} Df(x) \cdot f(x) + \int_0^1 \frac{dg}{d\mu} \, d\mu$$

It can be shown that the last term is $O(t^3)$ ▨.

EXERCISE

Use this to prove the composition formula (of Lie, Trotter, ...)

$$\Phi_g^{-\varepsilon}(\Phi_f^{-\varepsilon}(\Phi_g^{\varepsilon}(\Phi_f^{\varepsilon}(x_0))))$$

$$= x_0 + \varepsilon^2 [f, g](x_0) + O(\varepsilon^3)$$

Hint: carry along all the way, terms up to ε^2, and refer all values $f(x)$, $g(x)$, $Df(x)$, $Dg(x)$, back to $x = x_0$. Again for this purpose use the fundamental theorem of integral calculus.