Lecture 1 ENEE 664 Spring 2004 P. S. Krishnaprasad

Consider the linear control system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1}$$

with $x(t_0) = x_0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $A(\cdot)$ and $B(\cdot)$ matrix-valued functions. If $A(t)$ has elements that are continuous functions of t on $[t_0, t_1]$ then the (transition) matrix

$$\Phi(t, t_0) = \lim_{k \to \infty} M_k(t, t_0) \qquad t_0 \le t \le t_1 \tag{2}$$

exists and satisfies the homogeneous differential equation

$$\begin{aligned} \dot{\Phi}(t, t_0) &= A(t)\,\Phi(t, t_0) \\ \Phi(t_0, t_0) &= \mathbb{1} \quad \text{the } n \times n \text{ identity matrix.} \end{aligned} \tag{3}$$

Here the M_k are defined recursively as,

$$\begin{aligned} M_0(t, t_0) &\equiv \mathbb{1} \\ M_k(t, t_0) &= \mathbb{1} + \int_{t_0}^{t} A(\sigma)\, M_{k-1}(\sigma, t_0)\, d\sigma \\ & \qquad k = 1, 2, 3, \ldots \qquad t_0 \le t \le t_1 \end{aligned} \tag{4}$$

leading to the series formula (due to Peano and Baker)

$$\begin{aligned} \Phi(t, t_0) = \mathbb{1} &+ \int_{t_0}^{t} A(\sigma)\, d\sigma + \int_{t_0}^{t} \int_{t_0}^{\sigma_1} A(\sigma_1) A(\sigma)\, d\sigma\, d\sigma_1 \\ &+ \int_{t_0}^{t} \int_{t_0}^{\sigma_1} \int_{t_0}^{\sigma_2} A(\sigma_1) A(\sigma_2) A(\sigma)\, d\sigma\, d\sigma_2\, d\sigma_1 \\ &+ \cdots \end{aligned} \tag{5}$$

which reduces to the usual series formula for $e^{A(t-t_0)}$

when $A(t) \equiv A$ a constant.

The transition matrix gives the solution

$$x(t) = \Phi(t,t_0)\, x_0$$

for the homogeneous differential equation

$$\dot{x}(t) = A(t)\, x(t)$$

obtained by setting $u(t) \equiv 0$ in (1). To solve (1) for general $u(\cdot)$, let $z(t) = \left(\Phi(t,t_0)\right)^{-1} x(t)$. Then

$$\dot{z}(t) = \frac{d}{dt}\left(\Phi(t,t_0)^{-1}\right) x(t) + \Phi(t,t_0)^{-1} \frac{d}{dt} x(t)$$

$$= -\Phi(t,t_0)^{-1} \frac{d\Phi}{dt}(t,t_0)\, \Phi(t,t_0)^{-1}\, x(t)$$

(6) $$+ \Phi(t,t_0)^{-1} \left(A(t)x(t) + B(t)u(t)\right)$$

$$= -\Phi^{-1} A \Phi \Phi^{-1} x + \Phi^{-1} A x + \Phi^{-1} B u$$

$$= \Phi^{-1}(t,t_0)\, B(t)\, u(t)$$

which is easily integrated to obtain

(7) $$z(t) = \int_{t_0}^{t} \Phi^{-1}(\sigma,t_0)\, B(\sigma)\, u(\sigma)\, d\sigma + z(t_0).$$

Recall $z(t_0) = \Phi(t_0,t_0)^{-1} x(t_0) = x(t_0) = x_0$.

Expressing x in terms of z we obtain

$$
\boxed{\begin{aligned}
x(t) &= \Phi(t,t_0)\,x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(\sigma,t_0)\,B(\sigma)\,u(\sigma)\,d\sigma \\
&= \Phi(t,t_0)\,x_0 + \int_{t_0}^{t} \Phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma
\end{aligned}} \tag{8}
$$

where we have used the easily verified properties

$$
\Phi(r,s)^{-1} = \Phi(s,r)
$$

$$
\Phi(t,s)\,\Phi(s,r) = \Phi(t,r)
$$

for any t, s, r (order immaterial).

Equation (8) is the famous <u>variation of constants formula</u> due to Lagrange.

§ 1.1. <u>Reachability / Accessibility Problem</u>

Given $t_1 > t_0$ and $x_1 \in \mathbb{R}^n$ does there exist a control $u(\cdot)$ that drives system (1) from $x(t_0) = x_0$ to $x(t_1) = x_1$? Usually one restricts the class of allowable controls to (piecewise) continuous, smooth etc.

Accessibility of x_1 is equivalent to solvability for $u(\cdot)$ of the equation

$$x_1 - \Phi(t_1, t_0)\, x_0 = \int_{t_0}^{t_1} \Phi(t_1, \sigma)\, B(\sigma)\, u(\sigma)\, d\sigma,$$

equivalently,

(9) $$L(u) \triangleq -\int_{t_0}^{t_1} \Phi(t_0, \sigma)\, B(\sigma)\, u(\sigma)\, d\sigma = x_0 - \Phi(t_0, t_1)\, x_1 .$$

Here $L : \mathcal{U} \to \mathbb{R}^n$ is a linear map which takes the vector space of (continuous) control functions $u : [t_0, t_1] \to \mathbb{R}^m$ into the vector space $\mathbb{R}^n$.

Thus (x_1, t_1) is accesible from (x_0, t_0) iff $(x_0 - \Phi(t_0, t_1)\, x_1) \in \mathcal{R}(L)$ the range space of L. The system (1) is accesible if $\mathcal{R}(L) = \mathbb{R}^n$.

Computing $\mathcal{R}(L)$ is made easy by the use of the concept of adjoint.

Given two vector spaces V and W equipped with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively, the adjoint of a linear map $L : V \to W$ is the linear map $L^* : W \to V$ defined by

$$\langle L^*(w), v \rangle_V = \langle w , L(v) \rangle_V$$

$\forall$ $v \in V$ and $w \in W$.

Concretely if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ then the Euclidean inner products on V and W define $L^* = L^T$ the transpose for L a matrix $: \mathbb{R}^n \to \mathbb{R}^m$. (often I denote transpose by a superscript prime $'$ instead of a superscript T).

<u>Claim:</u> $\mathcal{R}(L) = \mathcal{R}(LL^*)$

<u>Proof:</u> <u>Homework Exercise</u> 1.

This result is extremely useful when L is a linear map from an infinite dimensional space into a finite dimensional space. Then LL^* is representable as a matrix and it is easy to check what the range of LL^* is.

If $V = \mathcal{U} = \{ u : [t_0, t_1] \to \mathbb{R}^m : \text{continuous} \}$ and inner product on $\mathcal{U}$ is $\langle u_1(\cdot), u_2(\cdot) \rangle_{\mathcal{U}} = \int_{t_0}^{t_1} u_1'(\sigma)\, u_2(\sigma)\, d\sigma$, and $W = \mathbb{R}^n$ with inner product, $\langle x, y \rangle_W = \sum_{i=1}^{n} x_i y_i$ then for

(10) $$L(u) = - \int_{t_0}^{t_1} \Phi(t_0, \sigma)\, B(\sigma)\, u(\sigma)\, d\sigma$$

$L^*(x)$ is a function $\in \mathcal{U}$
$$L^*(x)(t) = - B'(t)\, \Phi(t_0, t)\, x$$

Then $LL^*: \mathbb{R}^n \to \mathbb{R}^n$ is given by

11) $$LL^* = W(t_0 t_1) = \int_{t_0}^{t_1} \Phi(t_0,\sigma) B(\sigma) B'(\sigma) \Phi'(t_0,\sigma)\, d\sigma$$

By the previous claim, $\mathcal{R}(L) = \mathcal{R}(W(t_0,t_1))$.

We give the proof of this special case.

proof: ($\Rightarrow$) Let $x \in \mathcal{R}(W(t_0 t_1))$, i.e. $\exists\, \eta \in \mathbb{R}^n$ s.t.

$$x = W(t_0,t_1)\eta$$

choose $u(t) = L^*\eta$. Then $L(u) = LL^*\eta = x$

by assumption. Hence $x \in \mathcal{R}(L)$.

Thus $\mathcal{R}(W(t_0 t_1)) \subseteq \mathcal{R}(L)$

($\Leftarrow$) Suppose $x_1 \notin \mathcal{R}(W)$. We wish to show that $x_1 \notin \mathcal{R}(L)$. We will do this by contradiction. So assume $x_1 \in \mathcal{R}(L)$.

Since $x_1 \notin \mathcal{R}(W)$, $\exists\ x_2 \in \mathrm{Ker}(W^*)$

s.t. $x_2' x_1 \neq 0$ [Fredholm alternative — see Homework 1]

But $L(u) = x_1$ for some $u \in \mathcal{U}$, by hypothesis.

Then $$x_2' \int_{t_0}^{t_1} \Phi(t_0,\sigma) B(\sigma) u(\sigma)\, d\sigma \neq 0$$

On the other hand $x_2 \in \mathrm{Ker}(W^*) = \mathrm{Ker}(W)$

because W is symmetric. $\Rightarrow$ $x_2' W x_2 = 0$

$$\Leftrightarrow \int_{t_0}^{t_1} x_2' \, \Phi(t_0,\sigma) B(\sigma) B'(\sigma) \Phi'(t_0,\sigma) x_2 \, d\sigma = 0$$

$$\Leftrightarrow \int_{t_0}^{t_1} \| B'(\sigma) \Phi'(t_0,\sigma) x_2 \|^2 \, d\sigma = 0$$

$$\Rightarrow \quad B'(\sigma) \Phi'(t_0,\sigma) x_2 = 0$$ by continuity of integrand.

$\Rightarrow$ contradiction !

Hence $\quad x_1 \notin \mathcal{R}(L)$.

From the two parts we have shown

$$\mathcal{R}(W) \subseteq \mathcal{R}(L)$$

$$\mathcal{R}(L) \subseteq \mathcal{R}(W)$$

$$\Rightarrow \quad \mathcal{R}(W) = \mathcal{R}(L) \qquad \blacksquare$$

The symmetric matrix W is called the <u>accessibility Gramian</u>.

<u>Definition</u> The system (1) is said to be accessible from (x_0,t_0) if <u>any</u> (x_1,t_1) $t_1 > t_0$ is accessible from (x_0,t_0).

From what we have shown accessibility of the <u>system</u> is equivalent to $\mathcal{R}(W) = \mathbb{R}^n$ i.e. W is invertible.

A control which drives (1) from (x_0, t_0) to (x_1, t_1) is given by

$$(12) \qquad u_0(t) = -B'(t)\,\Phi'(t_0, t)\,\eta_0$$

for η_0 such that

$$W(t_0, t_1)\,\eta_0 = x_0 - \Phi(t_0, t_1)\,x_1$$

If the system is accessible then W is invertible and $\eta_0 = W^{-1}(t_0, t_1) \cdot [x_0 - \Phi(t_0, t_1)\,x_1]$.

§ 1.2. <u>Optimality</u>

u_0 is better than any other control u_1 in the sense that

$$\int_{t_0}^{t_1} u_0'(t)\,u_0(t)\,dt \leq \int_{t_0}^{t_1} u_1'(t)\,u_1(t)\,dt$$

for $L(u_1) = x_0 - \Phi(t_0, t_1)\,x_1$.

proof:
$$L(u_1) = L(u_0) \Rightarrow L(u_1 - u_0) = 0.$$

$$\eta_0' L(u_1 - u_0) = 0$$

$$\leftrightarrow \quad \langle \eta_0, L(u_1 - u_0) \rangle_{\mathbb{R}^n} = 0$$

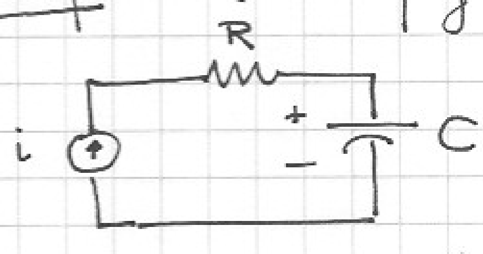
$$\leftrightarrow \quad \langle L^* \eta_0, u_1 - u_0 \rangle_{\mathcal{U}} = 0$$

(from definition of adjoint)

$$\leftrightarrow \quad \langle u_0, u_1 - u_0 \rangle_{\mathcal{U}} = 0$$

Then
$$0 \le \langle u_1 - u_0, u_1 - u_0 \rangle_{\mathcal{U}}$$
$$= \langle u_1, u_1 \rangle - \langle u_0, u_1 - u_0 \rangle - \langle u_1, u_0 \rangle$$
$$= \langle u_1, u_1 \rangle - \langle u_1, u_0 \rangle$$
$$= \langle u_1, u_1 \rangle - \langle u_1 - u_0 + u_0, u_0 \rangle$$
$$= \langle u_1, u_1 \rangle - \langle u_1 - u_0, u_0 \rangle - \langle u_0, u_0 \rangle$$
$$= \langle u_1, u_1 \rangle - \langle u_0, u_0 \rangle \qquad \blacksquare$$

Example (Charging a capacitor)



voltage across capacitance $= v$

capacitance $= C$

$$C \frac{dv}{dt} = i$$

minimize I^2R losses (// resistive) during charging from V_0 to V_1.

$$\longleftrightarrow \quad \min_{i} \int_0^{t_1} R\, i^2(t)\, dt$$

$v(0) = V_0$
$v(t_1) = V_1$

Here $A(t) \equiv 0$ $\quad B(t) \equiv \frac{1}{C}$ $\quad$ state $= v$

control $= i$

$$W(0, t_1) = \int_0^{t_1} \frac{1}{C^2}\, dt = \frac{t_1 - 0}{C^2} = \frac{t_1}{C^2}$$

$$\eta_0 = C^2 \, \frac{V_0 - V_1}{t_1}$$

$$i_0(t) = -\frac{1}{C}\, C^2 \frac{(V_0 - V_1)}{t_1}$$

$$= \frac{C\,(V_1 - V_0)}{t_1} = \text{constant}$$ (optimal charging current)

If $V_0 = 0$, a short, one can see that

over $[0, t_1]$

$$\text{efficiency} = \frac{\text{energy stored}}{\text{energy delivered}}$$

$$= \;?$$ ← compute this! $\square$

Show that

$$\frac{dW}{dt}(t,t_1) = A(t)\,W(t,t_1) + W(t,t_1)\,A'(t) - B(t)\,B'(t)$$

$$W(t_1,t_1) = 0$$

$$W(t_0,t_1) = W(t_0,t) + \Phi(t_0,t)\,W(t,t_1)\,\Phi'(t_0,t)$$

where W^{-1} exists

$$\frac{dW^{-1}(t,t_1)}{dt} = -A'(t)\,W^{-1}(t,t_1) - W^{-1}(t,t_1)\,A(t) + W^{-1}(t,t_1)\,B(t)\,B'(t)\,W^{-1}(t,t_1)$$

a Riccati type equation