Second Order Necessary Conditions

In calculus the Hessian enters through the discussion of second order necessary conditions on extrema.

Theorem: Let $f : U \to \mathbb{R}$ be a C^2 function defined on an open subset U of a normed vector space. Suppose f has a local minimum at $m \in U$. Then

$$D^2 f(m)\,(h, h) \geq 0 \qquad \forall\, h \in X$$

Proof Since $Df(m) = 0$, Taylor's formula gives

$$f(m+h) - f(m) = \frac{1}{2} D^2 f(m)(h,h) + r(h)$$

where the remainder satisfies

$$\lim_{h \to 0} \frac{r(h)}{\|h\|^2} = 0$$

< For a discussion of Taylor's formula see lecture notes 11 (b) >

Since f has a local minimum at m,

$$f(m+h) - f(m) \geq 0,$$

for $\|h\|$ small enough, say $\|h\| < \varepsilon$.

Thus for suitable $\varepsilon > 0$

$$D^2 f(m)\,(h,h) + 2\,r(h) \geq \qquad \text{for } \|h\| < \varepsilon.$$

Let $h \in X$ and take $\lambda \in \mathbb{R}$ so that $\lambda \neq 0$, a parameter, $\|\lambda h\| < \varepsilon$.

Since $D^2 f(m)$ is bilinear,

$$D^2 f(m)(\lambda h, \lambda h) = \lambda^2 D^2 f(m)(h, h)$$

Thus $\lambda^2 D^2 f(m)(h,h) + 2\gamma(\lambda h) \geq 0$.

Divide by λ^2 and let $\lambda \to 0$. It follows that

$$D^2 f(m)(h,h) \geq 0.$$

<u>Exercise</u> Suppose f is C^3. If $Df(m) = 0$, $D^2 f(m) = 0$ but $D^3 f(m) \neq 0$, then show that f <u>cannot</u> have a local minimum at m.

Using the Taylor's formula as above, it is possible to derive a <u>sufficient condition</u> for f to have a (strict) minimum at m. We need stronger hypotheses on the underlying space. We also need two additional basic lemmas.

<u>Lemma 1.</u> Let $f : U \subset X \to Y$ be a C^2 map at $a \in U$ an open subset of a normed vector space X, with values in normed vector space Y.

Then

$$\frac{\|f(a+u+v)-f(a+u)-f(a+v)+f(a)-D^2f(a)(u,v)\|}{(\|u\|+\|v\|)^2}$$

$$\longrightarrow 0 \qquad \text{as} \qquad u,v \longrightarrow 0.$$

In particular $D^2f(a)$ is bilinear symmetric form:

$$D^2f(a)(u,v) = D^2f(a)(v,u) \qquad \forall\, u,v \in X. \ \blacksquare$$

<u>Lemma 2</u>

Suppose m is a <u>nondegenerate critical point</u> of f (<u>defn</u>: nondegeneracy $\leftrightarrow$ $Df(m)=0$ <u>and</u> the map $h \to D^2f(m)(h,\cdot)$ is an isomorphism of X and X^*, where X is Banach) defined on U open subset of a Banach space X. Suppose $D^2f(m)(h,h)>0$ $\forall\, h\in X$, $h\neq 0$. Then there is a constant $c>0$ such that $D^2f(m)(h,h) \geq c\|h\|^2$ $\forall\, h\in X$.

<u>Proof</u>: Let $L = D^2f(m)$. L is non-degenerate by hypothesis. $x \mapsto L(x,\cdot)$ is an isomorphism onto X^*.

If M is the norm of the inverse isomorphism, then

$$\|x\|_X \leq M\,\|L(x,\cdot)\|_{X^*}.$$

$$\|L(x,\cdot)\|_{X^*} = \sup_{\|y\|=1} |L(x,y)|.$$ So, given x

there is a $y \in X$ with norm 1 s.t.

$$2\,|L(x,y)| \geq \|L(x,\cdot)\|_{X^*}$$

Thus,

$$\|x\| \leq 2M\,|L(x,y)|.$$

Symmetry of L (Lemma 1) and positivity hypothesis $\Rightarrow$

$$0 < L(x+ty, x+ty)$$

$$= t^2 L(y,y) + 2t\,L(x,y) + L(x,x) \qquad \forall\ t \in \mathbb{R}$$

Letting $\|L\|$ denote the norm of the continuous bilinear functional L

$$|L(x,y)|^2 \leq L(x,x)\,L(y,y) \qquad \text{(discriminant condition)}$$

$$\leq \|L\| \cdot L(x,x) \qquad (\because \|y\|=1)$$

Combining this with the estimate on $\|x\|$,

$$\|x\|^2 \leq 4M^2\,|L(x,y)|^2$$

$$\leq 4M^2\,\|L\| \cdot L(x,x)$$

$$\Rightarrow \quad L(x,x) \geq \frac{1}{4M^2}\frac{1}{\|L\|}\,\|x\|^2$$

$$= c\,\|x\|^2$$

With the help of lemma 2 we can prove

<u>Theorem</u> (sufficiency)

Let U be an open subset of a Banach space X. Let $f: U \to \mathbb{R}$ be C^2 at $m \in U$. If m is a <u>nondegenerate</u> <u>critical</u> <u>point</u> of f and if $D^2 f(m)(h,h) > 0 \quad \forall\, h \in X,\ h \neq 0$, then f has a strict minimum at m.

~~Proof~~ Taylor's formula and lemma 2 $\Rightarrow$

$$f(m+h) - f(m) = \frac{1}{2} D^2 f(m)(h,h) + r(h)$$

$$\geq \frac{c}{2} \|h\|^2 + r(h)$$

where $r(h) = o(\|h\|^2)$.

Thus we can find $\delta > 0$ s.t

$$\|h\| < \delta \Rightarrow |r(h)| \leq \frac{c}{4} \|h\|^2 .$$

If $\|h\| < \delta$ then

$$\begin{aligned} f(m+h) - f(m) &\geq \frac{c}{2} \|h\|^2 + r(h) \\ &\geq \frac{c}{2} \|h\|^2 - |r(h)| \\ &\geq \frac{c}{2} \|h\|^2 - \frac{c}{4} \|h\|^2 \\ &= c/2 \, \|h\|^2 \end{aligned}$$

Remark (a global result)

If X is a Hilbert space, and suppose f is C^3 on X. Suppose f has a unique critical point m and that this critical point is nondegenerate, and that $D^2 f(m)(h,h) > 0 \quad \forall\ h \in X,\ h \neq 0$.

Then $f(m) < f(x) \quad \forall\ x \in X,\ x \neq m$.