

An example on Fréchet derivatives.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ have continuous first partial derivatives $\frac{\partial f}{\partial x_i}$. Then $\Delta f(x; h) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i$ is also the Fréchet derivative $Df(x; h)$. \square

Clearly, the row vector $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ defines a bounded linear operator on \mathbb{R}^n . We need to show the limit property

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i\|}{\|h\|} = 0$$

Equivalently, we need to show that, given any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\|h\| < \delta$,

$$\frac{\|f(x+h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i\|}{\|h\|} < \epsilon,$$

Equivalently,

$$\|f(x+h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i\| < \epsilon \|h\|.$$

To see this, first observe that, from the hypothesis on continuity of first partial derivatives,

given $\epsilon > 0$, there exists $\delta_i > 0$ s.t.

$$y \in B_{\delta_i}^i(x) \iff \|y - x\| < \delta_i$$

$$\implies \left\| \frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x) \right\| < \frac{\epsilon}{n} \quad i=1, 2, \dots, n$$

Let $\delta = \min_{i=1,2,\dots,n} \delta_i$. Then,

$$y \in B_\delta(x) \Rightarrow \left\| \frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x) \right\| < \frac{\epsilon}{n} \quad \forall i=1,2,\dots,n.$$

Let e_1, e_2, \dots, e_n be the standard basis column vectors in \mathbb{R}^n with e_i having only the i th element nonzero = 1

Then $h = \sum_{i=1}^n h_i e_i$; define $g_0 = 0$ and
 $g_k = \sum_{i=1}^k h_i e_i \quad k=1,2,\dots,n$. Thus $g_n = h$.

Suppose $\|h\| < \delta$

NOTE: ON THE BOARD, ON THURSDAY FEB 26, I KEPT ON WRITING e_k 's AS ROW VECTORS. I SHOULD HAVE WRITTEN COLUMN VECTORS.

Observe that for \mathbb{R}^n with the norm $\|x\| = \sum_{i=1}^n |x_i|$,
 $\|g_k\| = \sum_{i=1}^k |h_i| \leq \sum_{i=1}^n |h_i| = \|h\| < \delta$

$$\begin{aligned} \text{Then } & \left| f(x+h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) h_i \right| \\ &= \left| \sum_{k=1}^n \left(f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k \right) \right| \\ &\leq \sum_{k=1}^n \left| f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k \right| \end{aligned}$$

$x+g_k = x+g_{k-1} + h_k e_k$. Below, assume $h_k > 0$ (by triangle inequality) for the moment.

Consider the function $F: [0, h_k] \rightarrow \mathbb{R}$

$$F(z) = f(x+g_{k-1} + z e_k)$$

$$\text{Then } F(h_k) = f(x+g_{k-1} + h_k e_k) = f(x+g_k)$$

Recall the mean value theorem MVT

Given $F: [a, b] \rightarrow \mathbb{R}$, F continuous on $[a, b]$
and differentiable on (a, b) , there is a ξ
 $a < \xi < b$

such that $F(b) - F(a) = F'(\xi) \cdot (b-a)$.

Apply MVT to $F: [0, h_k] \rightarrow \mathbb{R}$ defined on page 2.

Thus there exists α , $0 < \alpha < h_k$ such that

$$\begin{aligned} f(x+g_k) - f(x+g_{k-1}) &= f(x+g_{k-1} + h_k e_k) - f(x+g_{k-1}) \\ &= \frac{\partial f}{\partial x_k}(x+g_{k-1} + \alpha e_k) \cdot h_k \end{aligned}$$

Thus $f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k$

$$= \left(\frac{\partial f}{\partial x_k}(x+g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k}(x) \right) h_k$$

Clearly $\|(x+g_{k-1} + \alpha e_k) - x\| = \sum_{i=1}^{k-1} |h_i| + \alpha$

$$< \sum_{i=1}^k |h_i|$$
$$< \|h\| < \delta$$

Thus $x+g_{k-1} + \alpha e_k \in \mathcal{B}_\delta(x)$

$$\Rightarrow \left| \frac{\partial f}{\partial x_k}(x+g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k}(x) \right| < \frac{\varepsilon}{h}$$

~~Also in order to see that~~

$$\begin{aligned} \text{Thus } \left| f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k \right| &< \frac{\varepsilon}{h} |h_k| \\ &< \frac{\varepsilon}{h} \|h\| \end{aligned}$$

The assumption $h_k > 0$ can be replaced by $h_k < 0$ and interchanging b and a in MVT gives the same answer. ~~For~~ For $h_k = 0$ it is trivially true. So we find that

$$\begin{aligned}
 & |f(x+h) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i| \\
 & \leq \sum_{k=1}^n |f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k| \\
 & < \sum_{k=1}^n \frac{\varepsilon}{n} \|h\| \\
 & = \varepsilon \|h\|, \quad \text{whenever } \|h\| < \delta
 \end{aligned}$$

This completes the proof.

EXERCISE (due back in class ~~February~~ March 9, Tuesday)

Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $(x, t) \rightarrow g(x, t)$

have a continuous in x, t first partial

$$g'_x(x, t) \equiv \frac{\partial g}{\partial x}(x, t).$$

Let $f: C[0, 1] \rightarrow \mathbb{R}$

$$x \mapsto \int_0^1 g(x(t), t) dt$$

Then the Gateaux differential $Sf(x; h) = \int_0^1 \frac{\partial g}{\partial x}(x(t), t) h(t) dt$
 is also the Fréchet differential $Df(x; h)$. The norm

on $C[0, 1]$ is taken to be $\|x\| = \int_0^1 |x(t)| dt$.