Homework Set 7, ENEE 664, to be returned Thursday, 04/12/12.

1 Consider the optimal control problem:

$$\underset{u(\cdot)}{\text{Min}} \int_0^1 \left(u_1^2(t) + u_2^2(t)\right) dt$$

subject to

$$\dot{x}_1(t) = u_1(t)$$

$$\dot{x}_2(t) = u_2(t)$$

$$\dot{x}_3(t) = x_1^2(t)\, u_2(t) - x_2^2(t)\, u_1(t)$$

$$x_1(0) = x_1(1) = 0$$

$$x_2(0) = x_2(0) = 0$$

$$x_3(0) = 0 \qquad x_3(1) = 1$$

Show that this problem reduces to solving the anharmonic "oscillator" equation:

$$\ddot{w} + c_1 w + c_2 w^3 = 0$$

for suitable constants c_1 and c_2.

2 Consider the functional

$$g(x) = \int_0^1 \left(x_1(t)\,\dot{x}_2(t) - x_2(t)\,\dot{x}_1(t)\right) dt$$

defined on $X = \{x(\cdot) : [0, 1] \to \mathbb{R}^2 \mid x \text{ continuously differentiable } \}$ ~~and~~

with suitable norm

Define $\Omega = \{ x(\cdot) \in X \mid g(x) = 0, \; x_i(0) = 0; \; x_i(1) = 1 \}$, $i = 1, 2$

Under what conditions is a given $x \in \Omega$ a regular point of Ω?

3. Consider the problem:

Minimize $\int_0^1 \left(\frac{1}{2} \dot{x}(t)^2 + \frac{1}{2} x(t)^2 - t\, x(t) \right) dt$

$x(0) = 0$;
$x(1) = 1$

Does this problem have a unique extremal? If so, is it a minimum of J?

4. Show that a curve of minimum length ~~minimizing the length~~ joining two points on a sphere

$S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$

is on a great circle on the sphere

5. Find (the) point in $\mathbb{R}^2$ that is closest to the origin and also lies on the ellipse (i.e. A, B > 0)

$$\frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1.$$

Clearly state any second order conditions you use. You may refer to Professor Andre Tits' lecture notes for guidance.

6. Consider the problem of <u>minimizing</u>

$$J[x] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\, dt$$

with t_1 and t_2 given, and

$x(t_1) = x_1$, but $x(t_2)$ unspecified.

Derive first order necessary conditions for this problem.

7. It is desired to show that

$$\int_0^{\pi/2} (\dot{x}(t))^2\, dt \geq \int_0^{\pi/2} (x(t))^2\, dt$$

for $x(0) = 0$

Solve this problem using

(a) optimal control theory

(b) calculus of variations