Lecture 12 Maximum Principle & Necessary Conditions from the Calculus of Variations

The E-L equations and the Legendre condition can be interpreted in the following way via the Control Hamiltonian $\mathbb{H}$ (as different from mechanics)

Define $\mathbb{H} = \mathbb{H}(t, q, p, u)$

to be a function of time, state, costate, and control. as

$$\mathbb{H}(t, q, p, u) = p\, f(t, q, u) - L(t, q, u)$$

associated to the ~~pro~~ problem of optimal control,

$$\underset{u(\cdot)}{\text{Min}} \int_{t_1}^{t_2} L(t, q(t), u(t))\, dt$$

Subject to $\dot{q}(t) = f(t, q(t), u(t))$

Boundary Conditions. $q(t_1) = q_1$; $q(t_2) = q_2$ fixed.

In the special case $f(t, q, u) = u$, the optimal control problem is expressible as a problem of the calculus of variations and then the necessary conditions are:

$$\frac{d}{dt}\frac{\partial L}{\partial u}(t, q(t), \dot q(t)) - \frac{\partial L}{\partial q}(t, q(t), \dot q(t)) = 0 \qquad \text{(E–L)}$$

$$\frac{\partial^2 L}{\partial u^2}(t, q(t), \dot q(t)) \geq 0 \qquad \text{(Legendre)}$$

The control Hamiltonian in the special case is:

$$\mathbb{H}(t, q, p, u) = p u - L(t, q, u)$$

Observe:

$$\frac{\partial \mathbb{H}}{\partial p} = u$$

$$\frac{\partial \mathbb{H}}{\partial q} = -\frac{\partial L}{\partial q}$$

$$\frac{\partial \mathbb{H}}{\partial u} = p - \frac{\partial L}{\partial u}$$

$$\frac{\partial^2 \mathbb{H}}{\partial u^2} = -\frac{\partial^2 L}{\partial u^2}$$

Now, suppose $t \mapsto q(t)$ is a trajectory of the system $\dot q(t) = u(t)$ which satisfies the boundary condition and minimizes

$$\int_{t_1}^{t_2} L(t, q(t), u(t))\, dt \,.$$

Let $t \mapsto p(t) = \frac{\partial L}{\partial u}(t, q(t), \dot{q}(t))$. Then

by E-L,

$$\dot{q}(t) = u(t) = \frac{\partial \mathbb{H}}{\partial p}(t, q(t), p(t), \dot{q}(t))$$

$$\dot{p}(t) = \frac{d}{dt}\frac{\partial L}{\partial u}(t, q(t), \dot{q}(t))$$

(C)

$$= \frac{\partial L}{\partial q}(t, q(t), \dot{q}(t))$$

$$= -\frac{\partial \mathbb{H}}{\partial q}(t, q(t), p(t), \dot{q}(t)) \;;$$

by definition of $t \mapsto p(t)$

(M1) $$\frac{\partial \mathbb{H}}{\partial u}(t, q(t), p(t), \dot{q}(t)) = 0$$

by Legendre

(M2) $$\frac{\partial^2 \mathbb{H}}{\partial u^2}(t, q(t), p(t), \dot{q}(t)) \leq 0$$

Conditions (M1) and (M2) are necessary for

$$\mathbb{H}(t, q(t), p(t), \dot{q}(t)) = \max_{v} \mathbb{H}(t, q(t), p(t), v)$$

This suggests a conjecture / principle.

" Consider the optimal control problem

$$\min_{u(\cdot)} \int_{t_1}^{t_2} L(t, q(t), u(t))\, dt$$

subject to $\dot{q}(t) = f(t, q(t), u(t))$

BC $\left.\begin{array}{l} q(t_1) = q_1 \\ q(t_2) = q_2 \end{array}\right\}$ fixed

Associate the Hamiltonian

$$\mathbb{H} = \mathbb{H}(t, q, p, u)$$

$$= p\, f(t, q, u) - L(t, q, u).$$

Suppose the input-state pair

$$t \longmapsto (q^*(t), u^*(t))$$

solves the optimal control problem.

Then there exists a co-state trajectory

$t \longmapsto p(t)$ such that

$$\dot{q}^*(t) = \frac{\partial \mathbb{H}}{\partial p}(t, q^*(t), p(t), u^*(t))$$

$$\dot{p}(t) = -\frac{\partial \mathbb{H}}{\partial q}(t, q^*(t), p(t), u^*(t))$$

and

$$\cancel{\mathbb{H}(t, q^*(t), p(t), u^*(t)) = \max_{v} \mathbb{H}(t, q^*,}$$

$$\mathbb{H}(t, q^*(t), p(t), u^*(t))$$

$$= \max_{v} \mathbb{H}(t, q^*(t), p(t), v) \quad "$$

This is, essentially, modulo some technicalities on the space of control functions, the [illegible] co-workers).

See your own class notes for details of examples.

The statement above extends to multidimensional systems by letting

$$\mathbb{H}(t, q, p, u)$$

$$= p^T f(t, q, u) - L(t, q, u)$$