<u>An example on Fréchet derivatives.</u>

Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous first partial derivatives $\frac{\partial f}{\partial x_i}$. Then $\delta f(x; h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i$ is also the Fréchet derivative $Df(x;h)$. $\square$

Clearly, the row vector $\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ defines a bounded linear operator on $\mathbb{R}^n$. We need to show the limit property

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \|}{\| h \|} = 0$$

Equivalently, we need to show that, given any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\| h \| < \delta$,

$$\frac{\| f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \|}{\| h \|} < \varepsilon,$$

Equivalently,

$$\| f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \| < \varepsilon \| h \| .$$

To see this, first observe that, from the hypothesis on continuity of first partial derivatives,

given $\varepsilon > 0$, there exists $\delta_i > 0$ s.t.

$$y \in B_{\delta_i}(x) \Leftrightarrow \| y - x \| < \delta_i$$

$$\Rightarrow \| \frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x) \| < \frac{\varepsilon}{n} \qquad i = 1, 2, \ldots, n$$

Let $\delta = \min\limits_{i=1,2,\dots,n} \delta_i$. Then,

$$y \in B_\delta(x) \Rightarrow \left\| \frac{\partial f}{\partial x_i}(y) - \frac{\partial f}{\partial x_i}(x) \right\| < \frac{\varepsilon}{n} \qquad \forall\, i = 1,2,\dots,n.$$

Let $e_1, e_2, \dots, e_n$ be the standard basis <u>column</u> vectors in $\mathbb{R}^n$ with e_i having only the i^{th} element nonzero $= \underline{1}$

Then $h = \sum\limits_{i=1}^{n} h_i e_i$; define $g_0 = 0$ and

$$g_k = \sum_{i=1}^{k} h_k e_k \qquad k = 1,2,\dots,n. \quad \text{Thus } g_n = h.$$

<u>Suppose $\|h\| < \delta$</u>

<u>NOTE</u> : ON THE BOARD, ON THURSDAY FEB 26, I KEPT ON WRITING e_k's AS <u>ROW</u> VECTORS. I SHOULD HAVE WRITTEN <u>COLUMN</u> VECTORS.

Observe that for $\mathbb{R}^n$ with the norm $\|x\| = \sum\limits_{i=1}^{n} |x_i|$,

$$\|g_k\| = \sum_{k=1}^{k} |h_k| \le \sum_{i=1}^{n} |h_i| = \|h\| \quad \forall\, h.$$

Then
$$\left| f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \right|$$
$$= \left| \sum_{k=1}^{n} \left(f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x) h_k \right) \right|$$
$$\le \sum_{k=1}^{n} \left| f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k} h_k \right|$$

(by triangle inequality)

$x + g_k = x + g_{k-1} + h_k e_k$. Below, $\boxed{\text{assume } h_k > 0.}$ for the moment.

Consider the function $F : [0, h_k] \to \mathbb{R}$

$$F(z) = f(x + g_{k-1} + z e_k)$$

Then $F(h_k) = f(x + g_{k-1} + h_k e_k) = f(x + g_k)$

Recall the mean value theorem MVT

Given $F : [a, b] \to \mathbb{R}$, F continuous on $[a, b]$ and differentiable on (a, b), there is a ξ

$$a < \xi < b$$

such that
$$F(b) - F(a) = F'(\xi) \cdot (b - a).$$

Apply MVT to $F : [0, h_k] \to \mathbb{R}$ defined on page 2.

Thus there exists α, $0 < \alpha < h_k$ such that

$$\begin{aligned} f(x + g_k) &- f(x + g_{k-1}) \\ &= f(x + g_{k-1} + h_k e_k) - f(x + g_{k-1}) \\ &= \frac{\partial f}{\partial x_k}(x + g_{k-1} + \alpha e_k) \cdot h_k \end{aligned}$$

Thus
$$f(x+g_k) - f(x + g_{k-1}) - \frac{\partial f}{\partial x_k}(x)\, h_k$$
$$= \left(\frac{\partial f}{\partial x_k}(x + g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k}(x) \right) h_k$$

Clearly
$$\begin{aligned} \|(x + g_{k-1} + \alpha e_k) - x\| &= \sum_{i=1}^{k-1} |h_i| + \alpha \\ &< \sum_{i=1}^{k} |h_i| \\ &< \|h\| < \delta \end{aligned}$$

Thus
$$x + g_{k-1} + \alpha e_k \in B_\delta(x)$$
$$\Rightarrow \left| \frac{\partial f}{\partial x_k}(x + g_{k-1} + \alpha e_k) - \frac{\partial f}{\partial x_k}(x) \right| < \frac{\varepsilon}{n}$$

Thus
$$\begin{aligned} \left| f(x+g_k) - f(x + g_{k-1}) - \frac{\partial f}{\partial x_k}(x)\, h_k \right| &< \frac{\varepsilon}{n} |h_k| \\ &< \frac{\varepsilon}{n} \|h\| \end{aligned}$$

The assumption $h_k > 0$ can be replaced by $h_k < 0$ and interchanging b and a in MVT gives the same answer. For $h_k = 0$ it is trivially true. So we find that

$$|f(x+h) - f(x) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i|$$

$$\leqq \sum_{k=1}^{n} |f(x+g_k) - f(x+g_{k-1}) - \frac{\partial f}{\partial x_k}(x)\, h_k|$$

$$< \sum_{k=1}^{n} \frac{\varepsilon}{n} \|h\|$$

$$= \varepsilon \|h\|, \qquad \text{whenever} \quad \|h\| < \delta$$

This completes the proof.

EXERCISE (due back in class ~~Feb~~ March 9, Tuesday)

Let $g : \mathbb{R}^2 \to \mathbb{R}$

$$(x,t) \to g(x,t)$$

have a continuous in x, t first partial

$$g_x(x,t) \equiv \frac{\partial g}{\partial x}(x,t).$$

Let $f : C[0,1] \to \mathbb{R}$

$$x \mapsto \int_0^1 g(x(t), t)\, dt$$

Then the Gateaux differential $\delta f(x; h) = \int_0^1 \frac{\partial g}{\partial x}(x(t),t)\, h(t)\, dt$ is also the Fréchet differential $Df(x;h)$. The norm on $C[0,1]$ is the infinity norm

$$\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|$$

DiVOGA