The classical problem of the calculus of variations is that of finding a function x on an interval minimizing a functional of the form

$$J[x] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\,dt .$$

One does this by computing (around a candidate minimizer x) on a family of admissible variations h:

$$J[x+\varepsilon h] = \int_{t_1}^{t_2} L(t, x(t)+\varepsilon h(t), \dot{x}(t)+\varepsilon \dot{h}(t))\,dt$$

where $-\varepsilon_0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$ and determining whether, for every h, and for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$J[x+\varepsilon h] \geq J[x] \quad ?$$

A first order necessary condition is

$$\delta J(x; h) = 0 \qquad \forall\, h$$

More precisely, let $D[t_1, t_2]$ denote the space of functions $x(\cdot): [t_1, t_2] \to \mathbb{R}$, differentiable on $[t_1, t_2]$, and such that the derivative $\dot{x}(\cdot)$ is continuous on $[t_1, t_2]$.

We let $X = D[a, b]$ be a normed vector space with the norm

$$\|x\| = \max_{a \le t \le b} |x(t)| + \max_{a \le t \le b} |\dot{x}(t)|$$

Fixed End points

We restrict the functions $x(\cdot)$ to have fixed end points $x(t_1)$ and $x(t_2)$. Then the admissible variations h must satisfy $h(t_1) = h(t_2) = 0$.

Observe that the Grateaux differential

$$\delta J(x;h) = \frac{d}{d\alpha} J[x+\alpha h]\Big|_{\alpha=0}$$

(1) ...

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t,x,\dot{x})\, h(t) + \frac{\partial L}{\partial \dot{x}}(t,x,\dot{x})\, \dot{h}(t) \right) dt$$

EXERCISE Verify that (1) is also the Fréchet differential.

Theorem Suppose $L = L(t,x,\dot{x})$ has continuous first partial derivatives w.r.t. x and $\dot{x}$. Suppose

$J = J[x] = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\, dt$ has an extremum at x with fixed end points $x(t_1)$ and $x(t_2)$. Then,

$$\delta J[x;h] = 0 \quad \forall\, h \text{ such that } h(t_1) = h(t_2) = 0 \text{ and } h \in D[t_1,t_2],$$

equivalently

(2)
$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t,x,\dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t,x,\dot{x}) \right) h(t)\, dt$$

$$= 0 \quad \forall\, h \text{ such that } h(t_1) = h(t_2) = 0 \text{ and } h \in D[t_1,t_2]$$

equivalently,

(3)
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t,x,\dot{x}) = \frac{\partial L}{\partial x}.}$$
(Euler-Lagrange)

To show that (2) and (1) are the same, it appears that one should do an "integration by parts trick" to replace the term with $\dot{h}$ in (1) by one involving h. The catch is that we do not know that $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$ exists. This we do by a series of traditional lemmas.

Lemma 1

Suppose $\alpha(t)$ is a scalar function, continuous on $[t_1, t_2]$ and $\int_{t_1}^{t_2} \alpha(t) h(t)\, dt = 0 \quad \forall\, h(\cdot) \in D[t_1, t_2]$ with $h(t_1) = h(t_2) = 0$. Then $\alpha(t) \equiv 0$ on $[t_1, t_2]$.

Proof Assume $\alpha(t) \not\equiv 0$. Then there is a t^* in (t_1, t_2) such that $\alpha(t^*) \neq 0$. By continuity of α, there is an interval $[t_1', t_2'] \subset [t_1, t_2]$ and $t^* \in [t_1', t_2']$, such that,

$$\alpha(t) \neq 0 \quad \text{on} \quad [t_1', t_2']$$

and $\alpha(t)$ has the same sign throughout $[t_1', t_2']$, equal to $\operatorname{sgn}(\alpha(t^*))$.

Consider
$$h(t) = \begin{cases} (t - t_1')^2 (t_2' - t)^2 & t_1' \le t \le t_2' \\ 0 & \text{otherwise.} \end{cases}$$

which satisfies the hypotheses of the lemma. Also, we have,

$$\int_{t_1}^{t_2} \alpha(t) h(t)\, dt \;\; > 0 \quad (\text{if } \alpha(t^*) > 0)$$
$$< 0 \quad (\text{if } \alpha(t^*) < 0)$$

In either case we have a contradiction. Hence $\alpha(t) \equiv 0$ on $[t_1, t_2]$. ▨

Lemma 2 (DuBois Reymond) If α is continuous on $[t_1, t_2]$ and

$$\int_{t_1}^{t_2} \alpha(t)\, \dot{h}(t)\, dt = 0$$

for every $h \in \mathcal{D}[t_1, t_2]$ with $h(t_1) = h(t_2) = 0$.

Then $\alpha(t) \equiv$ a constant on $[t_1, t_2]$.

Proof: Let $c = \dfrac{1}{t_2 - t_1} \displaystyle\int_{t_1}^{t_2} \alpha(t)\, dt$. Then,

$$\int_{t_1}^{t_2} (\alpha(t) - c)\, dt = 0$$

Let $h(t) = \displaystyle\int_{t_1}^{t} (\alpha(\tau) - c)\, d\tau$. Clearly $\dfrac{dh}{dt} = \alpha(t) - c$ is continuous, and hence $h \in \mathcal{D}[t_1, t_2]$. Then,

$$\int_{t_1}^{t_2} (\alpha(\tau) - c)^2\, d\tau = \int_{t_1}^{t_2} (\alpha(\tau) - c)\, \dot{h}(\tau)\, d\tau$$

$$= \int_{t_1}^{t_2} \alpha(t)\, \dot{h}(t)\, dt - c\, h(t)\Big|_{t_1}^{t_2}$$

$$= 0 \qquad \text{by hypothesis}$$

$\Rightarrow \quad \alpha(t) \equiv c \qquad t \in [t_1, t_2]$. ▨

Lemma 3 If α and β are continuous on $[t_1, t_2]$ and $\displaystyle\int_{t_1}^{t_2} (\alpha(t) h(t) + \beta(t) \dot{h}(t))\, dt = 0$ for every $h \in \mathcal{D}[t_1, t_2]$ satisfying $h(t_1) = h(t_2) = 0$. Then

β is differentiable and $\dot\beta(t) = \alpha(t)$ in $[t_1, t_2]$.

<u>Proof</u> Define $\quad A(t) = \int_{t_1}^{t} \alpha(\tau)\, d\tau.$

Integration by parts yields,

$$\int_{t_1}^{t_2} \alpha(t)\, h(t)\, dt = A(t)\, h(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} A(\tau)\, \dot h(\tau)\, d\tau$$

$$= - \int_{t_1}^{t_2} A(\tau)\, \dot h(\tau)\, d\tau .$$

Hence $\displaystyle\int_{t_1}^{t_2} (\alpha(t) h(t) + \beta(t) \dot h(t))\, dt$

$$= \int_{t_1}^{t_2} (- A(t) \dot h(t) + \beta(t) \dot h(t))\, dt$$

$$= \int_{t_1}^{t_2} (\beta(t) - A(t))\, \dot h(t)\, dt$$

$$= 0 \qquad \forall\, h \in D[t_1, t_2],\ h(t_1) = h(t_2) = 0 \quad \text{(hypothesis)}$$

By continuity of $\beta(t) - A(t)$ and Lemma 2, we conclude that

$$\beta(t) - A(t) \equiv c \quad \text{a constant.}$$

$\Rightarrow \qquad \beta(t)$ exists and $\dot\beta(t) = \dot A(t)$

$$= \alpha(t) \qquad \boxtimes$$

<u>Remark</u>: The converses of all three lemmas above also hold. This is easy to see. $\square$

Returning to the statement of the theorem, it is clear from Lemmas 1–3 that we have shown that the integration by parts step is justified and (2) follows. Again applying Lemma 1, (3) follows. This completes the proof of the theorem. $\square$

The Euler-Lagrange equations have shaped the development of mathematics and physics in over two centuries. See course website for helpful links.

Letting $\gamma(t) = x^*(t)$ be a candidate extremum, the Euler-Lagrange equations say,

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}}\Big|_{t,\gamma(t),\dot\gamma(t)} + \frac{\partial}{\partial x}\frac{\partial L}{\partial \dot{x}}\Big|_{t,\gamma(t),\dot\gamma(t)}\dot\gamma + \frac{\partial}{\partial \dot{x}}\frac{\partial L}{\partial \dot{x}}\Big|_{t,\gamma(t),\dot\gamma(t)}\ddot\gamma = \frac{\partial L}{\partial x}\Big|_{t,\gamma(t),\dot\gamma(t)}$$

Thus the Euler-Lagrange equation are second order equations. They can be cast in first order form under certain conditions, yielding Hamilton's canonical equations.

<u>Hypothesis</u> Assume that the equation

$$p = \frac{\partial L}{\partial \dot{x}}$$

can be <u>inverted</u> for $\dot{x}$ to express $\dot{x}$ as a function of (t, x, p). $= F(t,x,p)$ $\square$

Definition Let $H = H(t,x,p) \triangleq \dot{x}p - L$

Hamiltonian

$$= F(t,x,p)\,p - L(t,x,F(t,x,p))$$

Then
$$\frac{\partial H}{\partial p} = F + \frac{\partial F}{\partial p}p - \left.\frac{\partial L}{\partial \dot{x}}\right|_{\dot{x}=F} \frac{\partial F}{\partial p}$$

$$= F + \frac{\partial F}{\partial p}p - p\frac{\partial F}{\partial p}$$

$$= F$$

$$= \dot{x}$$

$$\frac{\partial H}{\partial x} = \frac{\partial F}{\partial x}p - \frac{\partial L}{\partial x} - \left.\frac{\partial L}{\partial \dot{x}}\right|_{\dot{x}=F} \frac{\partial F}{\partial x}$$

$$= \frac{\partial F}{\partial x}p - \frac{\partial L}{\partial x} - p\frac{\partial F}{\partial x}$$

$$= -\frac{\partial L}{\partial x}$$

$$= -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \qquad \text{(from Euler–Lagrange)}$$

$$= -\frac{dp}{dt}$$

Thus we obtain Hamilton's canonical equations

(C) $$\boxed{\frac{dx}{dt} = \frac{\partial H}{\partial p}\ ; \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}}$$

Equivalence of Lagrangians

We will say that Lagrangians L_1 and L_2 are equivalent if they have the same set of extremals. Sometimes we are also interested in one-way implication i.e. when γ is an extremal of L_1 it is also an extremal of L_2.

here extremal $\leftrightarrow$ satisfies Euler Lagrange equation

Example Let $L = L(q, v) = b(q)(v, v) > 0$ for $v \neq 0$, be a quadratic form in the velocity $v = \dot{q}$ along curves γ. Every extremal γ of $\int_a^b L\, dt$ is also an extremal of $\int_a^b \sqrt{L}\, dt$.

Along γ, (letting $D_2 L = \frac{\partial L}{\partial q}$; $D_3 L = \frac{\partial L}{\partial v}$)

$$\frac{dL}{dt} = D_2 L \cdot \gamma' + D_3 L \cdot \gamma''$$

$$= \left[D_2 L - \frac{d}{dt} D_3 L \right] \cdot \gamma' + \frac{d}{dt} (D_3 L \cdot \gamma')$$

$$= 0 + \frac{d}{dt} (D_3 L \cdot \gamma') \qquad \text{(by Euler-Lagrange)}$$

By Leibnitz rule

$$D_3 L . v = D_3 [b(q)(v, v)] v$$

$$= 2\, b(q)(v, v)$$

$$= 2L$$

Thus $\frac{dL}{dt} = \frac{d}{dt}(2L) \Rightarrow \frac{dL}{dt} = 0.$

Now $\frac{d}{dt}(D_3 \sqrt{L}) - D_2(\sqrt{L})$

$$= \frac{d}{dt}\left(\frac{1}{2}\frac{1}{\sqrt{L}} D_3 L\right) - \frac{1}{2\sqrt{L}} D_2 L$$

$$= -\frac{1}{4} L^{-3/2} \frac{dL}{dt} D_3 L + \frac{1}{2} L^{-1/2}\left(\frac{d}{dt} D_3 L - D_2 L\right)$$

$$= 0 + 0$$ (since γ is an extremal of L)

These are the Euler Lagrange equations of $L^{1/2}$.
Thus γ is an <u>extremal</u> of $\sqrt{L}$.

<u>Remark</u> We have stated the results in this section of the notes, for the case of scalar $x(t)$. In fact the results hold when the $x(t)$ takes values in V, a normed linear space (possibly infinite dimensional), in which the norm is defined from a positive definite inner product, $\|x\| = \langle x, x\rangle$ and the space is <u>complete</u> in this norm (i.e. a <u>Hilbert Space</u>). Simply replace products such as $\alpha(t)\, h(t)$ by $\langle \alpha(t), h(t)\rangle$ etc. Easy to do for $V = \mathbb{R}^n$.