Lecture 7 ENEE 664, SPG 2004

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Free End Points and Transversality

Among all curves $t \mapsto x(t)$ starting at $x(t_1) = x_1$ at $t = t_1$ and ending at another curve $t \mapsto g(t)$ at unspecified final time and place, find one that minimizes $J = \int_{t_1}^{t_2} L(t, x(t), \dot{x}(t))\,dt$

Necessary Conditions

initial t_1 fixed $x(t_1) = x_1$

given target curve $t \mapsto g(t)$

admissible family of curves $t \mapsto x^{\varepsilon}(t)$

$$x^{\varepsilon}(t_2(\varepsilon)) = g(t_2(\varepsilon)) \qquad -a < \varepsilon < a$$

suppose $\varepsilon = 0$ is an extremum of J. Then

denote $x^{\varepsilon}(t)\Big|_{\varepsilon=0}$ as $x(t)$, and

let $t_2 = t_2(\varepsilon)\Big|_{\varepsilon=0}$

$$J^{\varepsilon} \triangleq J[x^{\varepsilon}] = \int_{t_1}^{t_2(\varepsilon)} L(t, x^{\varepsilon}(t), \dot{x}^{\varepsilon}(t))\,dt$$

necessary condition for local extremum at $\varepsilon = 0$ is $\dfrac{dJ^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} = 0$

But $\left.\frac{dJ^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = L(t, x(t_2), \dot{x}(t_2)) \cdot \tau$

$$+ \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t))\, h(t) + \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\, \dot{h}(t) \right\} dt$$

where,

$$\tau = \left.\frac{d}{d\varepsilon} t_2(\varepsilon)\right|_{\varepsilon=0}$$

$$h(t) = \left.\frac{d}{d\varepsilon} x^{\varepsilon}(t)\right|_{\varepsilon=0}$$

also $h(t_1) = 0$ since $x^{\varepsilon}(t_1) = x(t_1) = x_1$

$-a < \varepsilon < a$

Since the family of curves $x^{\varepsilon}(\cdot)$ is not restricted in any way other than
(a) differentiability w.r.t ε and t (b) $x^{\varepsilon}(t_1) = x_1$
(c) $x^{\varepsilon}(t_2(\varepsilon)) = g(t_2(\varepsilon))$, it is clear that τ is allowed to take all possible real values.
Also the function $h(t)$ can be any continuously differentiable function satisfying

$h(t_1) = 0$

and $h(t_2)$ restricted as shown below.

Since $\quad x^{\varepsilon}(t_2(\varepsilon)) = g(t_2(\varepsilon))$

$$\frac{d}{d\varepsilon} x^{\varepsilon}(t_2(\varepsilon)) = \frac{d}{d\varepsilon} g(t_2(\varepsilon))$$

But l.h.s

$$= \left.\frac{d x^{\varepsilon}(t)}{d\varepsilon}\right|_{t=t_2(\varepsilon)} + \left.\frac{d x^{\varepsilon}(t)}{dt}\right|_{t=t_2(\varepsilon)} \frac{d t_2(\varepsilon)}{d\varepsilon}$$

r.h.s $= \left.\dfrac{dg}{dt}\right|_{t=t_2(\varepsilon)} \dfrac{d t_2(\varepsilon)}{d\varepsilon}$

Setting $\varepsilon = 0$ we get

$$h(t_2) + \dot{x}(t_2)\cdot\tau = \dot{g}(t_2)\,\tau$$

$$\Rightarrow \quad h(t_2) = (\dot{g}(t_2) - \dot{x}(t_2))\cdot\tau$$

<u>Invoking</u> the condition $\left.\dfrac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0$ for

the choice $\tau = 0$

$$\Rightarrow \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)]\, h(t) + \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\, \dot{h}(t))\, dt$$

$$= 0$$

$\forall\ h$ continuously differentiable on $[t_1, t_2]$ and $h(t_1)$ and $h(t_2) = 0$

By the three Lemmas

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \ .} \qquad (E-L)$$

This allows us to use integration by parts and conclude that

$$\left.\frac{dJ^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = L\,(t, x(t_2), \dot{x}(t_2)) \cdot \tau$$

$$+ \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right) dt$$

$h(t)$

$$+ \left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_2} h(t_2)$$

$$- \left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_2} h(t_1)$$

$$= \left[L(t, x(t_2), \dot{x}(t_2)) + \frac{\partial L}{\partial \dot{x}}(t, x(t_2), \dot{x}(t_2)) \cdot (\dot{g}(t_2) - \dot{x}(t_2))\right] \tau$$

$$+ \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right] h(t)\, dt$$

But by E-L this reduces to

$$\left.\frac{dJ^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = \left[L + \frac{\partial L}{\partial \dot{x}}(\dot{g} - \dot{x})\right]_{t=t_2} \cdot \tau$$

$= 0 \quad \forall \tau$

$$\Rightarrow \quad (\text{Tran}) \quad \boxed{L(t_2, x(t_2), \dot{x}(t_2)) + \frac{\partial L}{\partial \dot{x}}(t_2, x(t_2), \dot{x}(t_2))\,(\dot{g}(t_2) - \dot{x}(t_2)) = 0}$$

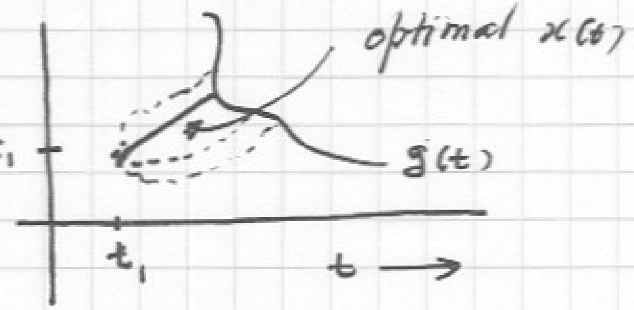
This new condition is added on to E-L, and it arises from the freedom in the final time t_2 and final state $x(t_2)$.

This important condition is known as the <u>transversality condition</u> and arises when the calculus of variations problem is of the "point-to-set" kind as opposed to "point-to-point" kind.

Example

$$L = \sqrt{1+\dot{x}^2}$$

g specified



minimizing the distance from a point to a specified curve

(E-L)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{1}{2}\frac{2\dot{x}}{\sqrt{1+\dot{x}^2}} - 0$$

$$= \frac{\sqrt{1+\dot{x}^2}\,\ddot{x} - \dot{x}\,(1+\dot{x}^2)^{-1/2}\,\frac{1}{2}\,2\dot{x}\ddot{x}}{(1+\dot{x}^2)}$$

$$= \frac{(1+\dot{x}^2)\ddot{x} - \dot{x}^2\ddot{x}}{(1+\dot{x}^2)^{3/2}}$$

$$= \frac{\ddot{x}}{(1+\dot{x}^2)^{3/2}} = 0$$

$\Rightarrow \quad \ddot{x} = 0 \quad \Rightarrow$ straight line.

Transversality Condition

$$\left(L + \frac{\partial L}{\partial \dot{x}}(\dot{g} - \dot{x}) \right)_{t=t_2}$$

$$= \sqrt{1+\dot{x}^2(t_2)} + \frac{\dot{x}(t_2)}{\sqrt{1+\dot{x}^2(t_2)}}\left(\dot{g}(t_2) - \dot{x}(t_2)\right)$$

$$= 0$$

$$\Leftrightarrow \quad 1 + \dot{x}^2(t_2) + \dot{x}(t_2)\dot{g}(t_2) - \dot{x}^2(t_2)$$

$$= 1 + \dot{x}(t_2)\dot{g}(t_2)$$

$$= 0$$

$$\Leftrightarrow \quad \dot{x}(t_2) = -\frac{1}{\dot{g}(t_2)}$$