Lecture 7 addendum.

Think of x as a vector in $\mathbb{R}^n$

Think of $g : \mathbb{R}^n \to \mathbb{R}$ smooth enough.

Trajectory $x(t)$ should end in $g(x) = 0$.

$$x(t_1) = x_1$$

family: $t \mapsto x^\varepsilon(t)$ $\quad -a < \varepsilon < a$

$$g(x^\varepsilon(t_2(\varepsilon))) = 0$$

Let $t_2 \triangleq t_2(\varepsilon)\big|_{\varepsilon=0}$

$$J^\varepsilon = J[x^\varepsilon]$$

$$= \int_{t_1}^{t_2(\varepsilon)} L\left(t, x^\varepsilon(t), \dot{x}^\varepsilon(t)\right) dt$$

necessary: $\frac{d}{d\varepsilon} J^\varepsilon\Big|_{\varepsilon=0} = 0$

$$\frac{dJ^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0} = L(t_2, x(t_2), \dot{x}(t_2))\, \tau$$

$$+ \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) \cdot h(t) + \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) \cdot \dot{h}(t) \right) dt$$

dot product

dot product

where $\tau = \frac{d}{d\varepsilon} t_2(\varepsilon)\Big|_{\varepsilon=0}$

$$h(t) = \frac{d}{d\varepsilon} x^\varepsilon(t)\Big|_{\varepsilon=0}$$

$h(t_1) = 0$ since $x^\varepsilon(t_1) = x(t_1) = x_1$
$-a < \varepsilon < a$

τ can take all possible real values.
$h(t_1) = 0$

what about $h(t_2)$?

$g(x^\varepsilon(t_2(\varepsilon))) = 0$ end point constraint

$$\left.\frac{\partial g}{\partial x}\right|_{x = x^\varepsilon(t_2(\varepsilon))} \cdot \left.\frac{d x^\varepsilon(t)}{d\varepsilon}\right|_{t = t_2(\varepsilon)} + \left.\frac{\partial g}{\partial x}\right|_{x = x^\varepsilon(t_2(\varepsilon))} \cdot \left.\frac{d}{dt} x^\varepsilon(t)\right|_{t = t_2(\varepsilon)} \cdot \frac{d t_2(\varepsilon)}{d\varepsilon}$$

$$= 0 \qquad -a < \varepsilon < a$$

Set $\varepsilon = 0$

$$\left.\frac{\partial g}{\partial x}\right|_{x = x(t_2)} \cdot \left(h(t_2) + \dot{x}(t_2)\, \tau \right) = 0$$

$\tau = 0$ and $h(t_2) = 0$ satisfies this

Now invoke $\tau = 0$ and see what $\left.\frac{d}{d\varepsilon} J^\varepsilon\right|_{\varepsilon = 0} = 0$ yields:

(i) $$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) \cdot h(t) + \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) \cdot \dot{h}(t) \right) dt$$

$= 0 \quad \forall\, h$ continuously diff. on $[t_1, t_2]$

and $\left. \begin{matrix} h(t_1) = 0 \\ h(t_2) = 0 \end{matrix} \right) \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \quad t_1 \le t \le t_2.$

Now it is OK to use integration by parts

$$\left.\frac{dJ^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = L(t_2, x(t_2), \dot{x}(t_2)) \cdot \tau$$

$$+ \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) \right) \cdot h(t)\, dt$$

$$+ \left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_2} \cdot h(t_2)$$

$$- \left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_1} \cdot h(t_1)$$

With E-L ~~&~~ and $\varepsilon = 0$ we get,

$$0 = L(t_2, x(t_2), \dot{x}(t_2))\,\tau$$

$$+ \left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_2} \cdot h(t_2)$$

and recall from previous page.

$$0 = \left.\frac{\partial g}{\partial x}\right|_{x=x(t_2)} \cdot \left(h(t_2) + \dot{x}(t_2)\,\tau\right)$$

which hold $\forall\, \tau$.

Setting $\tau = 0$ we get

$\Rightarrow$ $h(t_2)$ satisfies $\left.\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right|_{t=t_2} \cdot h(t_2) = 0$

and $\left.\frac{\partial g}{\partial x}\right|_{x=x(t_2)} \cdot h(t_2) = 0$

This imposes a constraint on $h(t_2)$.