Fixed Points

Theorem: Let X be a Banach space and let $S \subset X$ be a closed subset. Let $T: S \to S$ be a contraction map, i.e. there is a number ρ, $0 \le \rho < 1$ such that

$$\|T(x) - T(y)\| \le \rho \|x - y\| \quad \forall\, x, y \in S.$$

Then there is a unique $x^* \in S$ such that $x^* = T(x^*)$ (i.e. x^* is a fixed point of T).

Further this fixed point can be obtained as the limit of a sequence of successive approximations (Banach iterations).

Proof: Let $x_1 \in S$. Define the sequence $\{x_n : n \ge 1\} \subset S$ ~~Since $T: S \to S$, each~~ by $x_{n+1} = T(x_n)$.

Then,

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|T(x_k) - T(x_{k-1})\| \\
&\le \rho \|x_k - x_{k-1}\| \\
&\le \rho^2 \|x_{k-1} - x_{k-2}\| \quad \text{(repeating previous step)} \\
&\ \vdots \\
&\le \rho^{k-1} \|x_2 - x_1\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{k+r} - x_k\| &= \Big\| \sum_{i=1}^{r} (x_{k+i} - x_{k+i-1}) \Big\| \\
&\le \sum_{i=1}^{r} \|x_{k+i} - x_{k+i-1}\|
\end{aligned}$$

$$\leq \sum_{i=1}^{r} \rho^{k+i-2} \, \| x_2 - x_1 \|$$

$$\leq \rho^{k-1} \| x_2 - x_1 \| \sum_{i=0}^{\infty} \rho^{i}$$

$$= \frac{\rho^{k-1}}{1-\rho} \| x_2 - x_1 \| \longrightarrow 0 \quad \text{as } k \to \infty$$

$$(\because \rho < 1)$$

$\Rightarrow \{x_k\}$ is a Cauchy sequence. Since X is a Banach space, there exists $x^* \in X$ such that

$x^* = \lim_{n \to \infty} x_n$. But $\{x_n\} \subset S$. a closed set

$$\Rightarrow x^* \in S.$$

$$\| x^* - T(x^*) \| = \| x^* - x_n + x_n - T(x^*) \|$$

$$\leq \| x^* - x_n \| + \| x_n - T(x^*) \|$$

$$= \| x^* - x_n \| + \| T(x_{n-1}) - T(x^*) \|$$

$$\leq \| x^* - x_n \| + \rho \, \| x_{n-1} - x^* \|$$

$$\longrightarrow 0 \quad \text{as } n \to \infty$$

Hence $\| x^* - T(x^*) \| = 0 \Rightarrow x^* = T(x^*)$.

Suppose there is a $y^* \in S$ also a fixed point of T.

$$\| x^* - y^* \| = \| T(x^*) - T(y^*) \|$$

$$\leq \rho \, \| x^* - y^* \|$$

$\rho < 1 \Rightarrow \| x^* - y^* \| = 0 \Rightarrow x^* = y^*$.

We have shown uniqueness. ☐

If instead of a specific map T, we have a parametrized family of maps, then we can show continuity of the fixed point w.r.t. parameter θ. For such a family $T: \Theta \times S \to S$, define

$$T_\theta : S \to S \quad \text{by} \quad T_\theta(x) \triangleq T(\theta, x) \qquad x \in S$$

and

$$T^x : \Theta \to S \quad \text{by} \quad T^x(\theta) \triangleq T(\theta, x) \qquad \theta \in \Theta .$$

These maps T_θ and T^x are called <u>partial maps</u> associated to the family T.

<u>Theorem 2</u>: Let Θ be a <u>metric space</u> with metric d. Let X be a Banach space and let $S \subset X$ be a closed subset such that the family $T: \Theta \times S \to S$ has the following properties

(i) Each partial map $T_\theta : S \to S$, $\theta \in \Theta$, is a contraction with contraction coefficient $\rho < 1$

(ii) Each partial map $T^x : \Theta \to S$, $x \in S$, is continuous, i.e. given $\epsilon > 0$, there exists $\delta_x > 0$ such that,

$$d(\theta, \theta') < \delta_x \Rightarrow \| T^x(\theta) - T^x(\theta') \| < \epsilon .$$

Then the map $F : \Theta \to S$, $F(\theta) \triangleq x^*_\theta$ = unique fixed point of T_θ, is continuous.

<u>Proof</u>:
$$\begin{aligned}
\| x^*_\theta - x^*_{\theta'} \| &= \| T_\theta(x^*_\theta) - T_{\theta'}(x^*_{\theta'}) \| \\
&\le \| T_\theta(x^*_\theta) - T_\theta(x^*_{\theta'}) \| + \| T_\theta(x^*_{\theta'}) - T_{\theta'}(x^*_{\theta'}) \| \\
&\le \rho \, \| x^*_\theta - x^*_{\theta'} \| + \| T_\theta(x^*_{\theta'}) - T_{\theta'}(x^*_{\theta'}) \| \\
&\le \rho \, \| x^*_\theta - x^*_{\theta'} \| + \| T^{x^*_{\theta'}}(\theta) - T^{x^*_{\theta'}}(\theta') \|
\end{aligned}$$

$$\Rightarrow \quad \| x^*_\theta - x^*_{\theta'} \| \le \frac{1}{1-\rho} \| T_{x^*_{\theta'}}(\theta) - T_{x^*_{\theta'}}(\theta') \|$$

$$< \frac{\varepsilon}{1-\rho} \quad \text{whenever } d(\theta, \theta') < \delta_{x^*_{\theta'}}$$

[illegible]

Example 1 (Jacobi's algorithm)

Consider the system of linear equations,

$$Ax = b,$$

where A is a square matrix. We can write this as a fixed point problem

$$x = -D^{-1}(L+U)x + D^{-1}b$$

where $A = L + D + U$ denotes the decomposition of A into strictly lower triangular, diagonal and strictly upper triangular parts and we assume D is invertible.

Jacobi's algorithm to solve this problem:

$$x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b$$

is a special case of Banach iteration for the map $T(x) = -D^{-1}(L+U)x + D^{-1}b$

Suppose A is diagonally dominant:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \qquad i = 1, 2, \dots, n$$

Then $T : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction.

What is ρ in this case?

Example 2 Consider the scalar equation

$g(x) = x^2 - b$ where $b > 0$.

Let $y = 1 - x$. Finding the (positive) square root of b is a fixed point problem,

$$y = \frac{1}{2}\left((1-b) + y^2\right) = T(y)$$

Suppose $|1-b| < \rho < 1$.

Then T maps the closed subset $S = \{y : |y| \le \rho\} \subset \mathbb{R}$ into itself and it is a contraction on S with parameter ρ.

Thus the algorithm (Banach iteration)

$$y_{n+1} = \frac{1}{2}\left((1-b) + y_n^2\right)$$

Converges for $|1-b| < \rho < 1$

It is equivalent to

$$x_{n+1} = x_n - \frac{1}{2} x_n^2 + \frac{1}{2} b$$

How does it compare with Newton's method?