§1.5 Free end-point problems

Consider the problem of minimizing

$$\eta = \int_{t_0}^{t_1} \left(x'(t) L(t) x(t) + u'(t) u(t) \right) dt + x'(t_1) Q x(t_1)$$

along trajectories of $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ and $x(t_0) = x_0$. Without loss of generality assume that $L(t) = L'(t)$ and $Q = Q'$.

η contains a trajectory cost, a control cost and a terminal cost.

One can find an optimal control u_0 for this free end-point problem by a completion of squares trick. This involves the Riccati equation

$$\dot{K}(t) = -A'(t)K(t) - K(t)A(t) + K(t)B(t)B'(t)K(t) - L(t)$$

Everything we do hinges on the

Fundamental Lemma (path independent integrals)

Given (x, u) satisfying

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$$x'(t_1) K(t_1) x(t_1) - x'(t_0) K(t_0) x(t_0) = \int_{t_0}^{t_1} \begin{pmatrix} u'(t) & x'(t) \end{pmatrix} \begin{bmatrix} 0 & B'(t)K(t) \\ K(t)B(t) & \dot{K}(t) + A'(t)K(t) + K(t)A(t) \end{bmatrix} \begin{pmatrix} u(t) \\ x(t) \end{pmatrix} dt$$

Proof: For the statement of the lemma to make sense, we only need $K(t)$ to be any symmetric matrix valued function on $[t_0, t_1]$ with a continuous derivative $\dot{K}$ on (t_0, t_1).

$$\text{l.h.s.} = \int_{t_0}^{t_1} \frac{d}{dt}\left(x'(t)\, K(t)\, x(t)\right) dt$$

$$= \int_{t_0}^{t_1} \left(\dot{x}' K x + x' \dot{K} x + x' K \dot{x}\right) dt$$

$$= \int_{t_0}^{t_1} \left((Ax+Bu)' K x + x' \dot{K} x + x' K (Ax+Bu)\right) dt$$

$$= \text{r.h.s.}$$

Remark We use it below in the form

$$0 = \int_{t_0}^{t_1} (u', x') \begin{pmatrix} 0 & B'K \\ KB & \dot{K}+A'K+KA \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt$$

$$+ \; x'(t_0)\, K(t_0)\, x(t_0) \; - \; x'(t_1)\, K(t_1)\, x(t_1)$$

and we add this 0 to "anything & everything".

Theorem 1 Let $A(\cdot)$, $B(\cdot)$, $L(\cdot) = L'(\cdot)$ and $Q = Q'$ be given. Suppose that there exists on the interval $[t_0, t_1]$ a solution $\Pi = \Pi(t, Q, t_1)$ of the Riccati equation

$$\dot{K} = -A'K - KA + KBB'K - L$$

$$K(t_1) = Q.$$

Then there exists a control u which minimizes

$$\eta = \int_{t_0}^{t_1} \left[u'(t)\, u(t) + x'(t)\, L(t)\, x(t)\right] dt + x'(t_1)\, Q\, x(t_1)$$

for the system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$; $x(t_0) = x_0$.

The minimum value of η is given by $x_0' \Pi(t_0, Q, t_1) x_0$.

The minimizing control in feedback or closed loop form is

$$u_0(t) = -B'(t)\Pi(t, Q, t_1)x(t).$$

In open loop form

$$u_0(t) = -B'(t)\Pi(t, Q, t_1)\Phi_{A-BB'\Pi}(t, t_0)x_0$$

where $\Phi_{A-BB'\Pi}(t, t_0)$ is the transition matrix for the system $\dot{x}(t) = \big(A(t) - B(t)B'(t)\Pi(t, Q, t_1)\big)x(t)$.

<u>Proof</u> From the fundamental lemma

$$\begin{aligned}
\eta &= \eta + 0 \\
&= \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt + x'(t_1) Q x(t_1) \\
&\quad + x_0' \Pi(t_0, Q, t_1) x_0 - x'(t_1) Q x(t_1) \\
&\quad + \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 0 & B'\Pi \\ \Pi B & \dot{\Pi} + A'\Pi + \Pi A \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt \\
&= \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 1 & B'\Pi \\ \Pi B & \Pi B B' \Pi \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt + x_0' \Pi(t_0, Q, t_1) x_0 \\
&= \int_{t_0}^{t_1} \| u + B'\Pi x \|^2 \, dt + x_0' \Pi(t_0, Q, t_1) x_0
\end{aligned}$$

Clearly for the choice $u = u_0 = -B'\Pi x$ $\eta_{min} = x_0' \Pi(t_0, Q, t_1) x_0$. ▨

Remark, here $\| \cdot \|$ stands for the Euclidean norm

$$\| x \| = (x' x)^{1/2}$$

Theorem 2 (special case of $L \equiv 0$)

Let $A(\cdot)$, $B(\cdot)$, $Q = Q'$ be given. Let W be the accessibility Gramian for the system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t); \qquad x(t_0) = x_0 .$$

If the matrix

$$H(t, t_1) = W(t, t_1) + \Phi(t, t_1)\, Q^{-1}\, \Phi'(t, t_1)$$

is invertible on $[t_0, t_1]$, then there exists a control which minimizes

$$\eta = \int_{t_0}^{t_1} u'(t)\,u(t)\,dt + x'(t_1)\,Q\,x(t_1),$$

and it takes the form

$$u_0(t) = -B'(t)\, H(t, t_1)^{-1}\, x(t).$$

Proof: Set $L \equiv 0$ in Theorem 1 and the Riccati equation becomes

$$\dot{K} = -A'K - KA + KBB'K; \qquad K(t_1) = Q$$

$M = K^{-1}$ satisfies

$$\dot{M} = MA' + AM - BB' \qquad \text{(Lyapunov equation)}$$

which is also satisfied by $W(t, t_1)$ (see Lecture 1 notes, page 11), and has the solution

$$H(t, t_1) = [W(t, t_1) + \Phi(t, t_1)\, Q^{-1}\, \Phi'(t, t_1)].$$

(use matrix variation of constants formula).

Then if H^{-1} exists, it satisfies the Riccati equation and we apply Theorem 1 to get the rest.

and the boundary condition $K(t_1) = H(t_1, t_1)^{-1}$
$$= (Q^{-1})^{-1} = Q$$

∎

§ 1.6 <u>Fixed end-point problems</u>

<u>Theorem 3</u> Assume that there exists a symmetric matrix K_1 such that the solution $\Pi(t, K_1, t_1)$ of the matrix Riccati equation

$$\dot{K} = -A'K - KA + KBB'K - L$$

exists on $[t_0, t_1]$. Then a differentiable trajectory $x(t)$ on $[t_0\ t_1]$ of the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \qquad x(t_0) = x_0, \quad \text{and} \quad x(t_1) = x_1$$

minimizes

$$\eta = \int_{t_0}^{t_1} \left[u'(t)u(t) + x'(t)L(t)x(t)\right] dt$$

<u>if and only if</u> it minimizes

$$\eta_1 = \int_{t_0}^{t_1} v'(t)v(t)\, dt$$

for the system

$$\dot{x}(t) = (A(t) - B(t)B'(t)\Pi(t, K_1, t_1))x(t) + B(t)v(t)$$

$x(t_0) = x_0$; and $x(t_1) = x_1$. Moreover along any trajectory satisfying the boundary conditions

$$\eta = \eta_1 + x_0'\Pi(t_0, K_1, t_1)x_0 - x_1'K_1x_1 .$$

Proof From the fundamental lemma on path-independent integrals,

$$\eta = \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt$$

$$+ \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 0 & B'\Pi \\ \Pi B & \dot{\Pi} + A'\Pi + \Pi A \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt$$

$$+ \; x_0' \, \Pi(t_0, K_1, t_1) \, x_0 - x_1' K_1 x_1$$

$$= \int_{t_0}^{t_1} (u' \; x') \begin{pmatrix} 1 & B'\Pi \\ \Pi B & \Pi B B'\Pi \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix} dt + x_0' \Pi(t_0, K_1, t_1) x_0 - x_1' K_1 x_1$$

$$= \int_{t_0}^{t_1} \| u(t) + B'(t) \cdot \Pi(t, K_1, t_1) \, x(t) \|^2 \, dt + x_0' \Pi(t_0, K_1, t_1) x_0 - x_1' K_1 x_1$$

Let $u(t) + B'(t) \Pi(t, K_1, t_1) x(t) = v(t)$

$$\Rightarrow \quad \eta = \eta_1 + x_0' \Pi(t_0, K_1, t_1) x_0 - x_1' K_1 x_1$$

To minimize η we have to minimize η_1 (w.r.t. $v(t)$)

This completes the proof ▨

Remark $v(t) \equiv 0$ does not work in general since it may not get us to (x_1, t_1).

So how do we compute optimal v?

From Lecture 1 page 8.

$$v_0(t) = -B'(t)\,\Phi'_{A-BB'\Pi}(t_0, t)\,\xi$$

where ξ is such that

$$\tilde{W}(t_0, t_1)\,\xi = x_0 - \Phi_{A-BB'\Pi}(t_0, t_1)\,x_1$$

and

$\tilde{W}(t_0, t_1)$ is the accessibility Gramian

for the system

$$\dot{x}(t) = \left(A(t) - B(t)B'(t)\Pi(t, K_1, t_1)\right)x(t) + B(t)\,v(t)$$

and

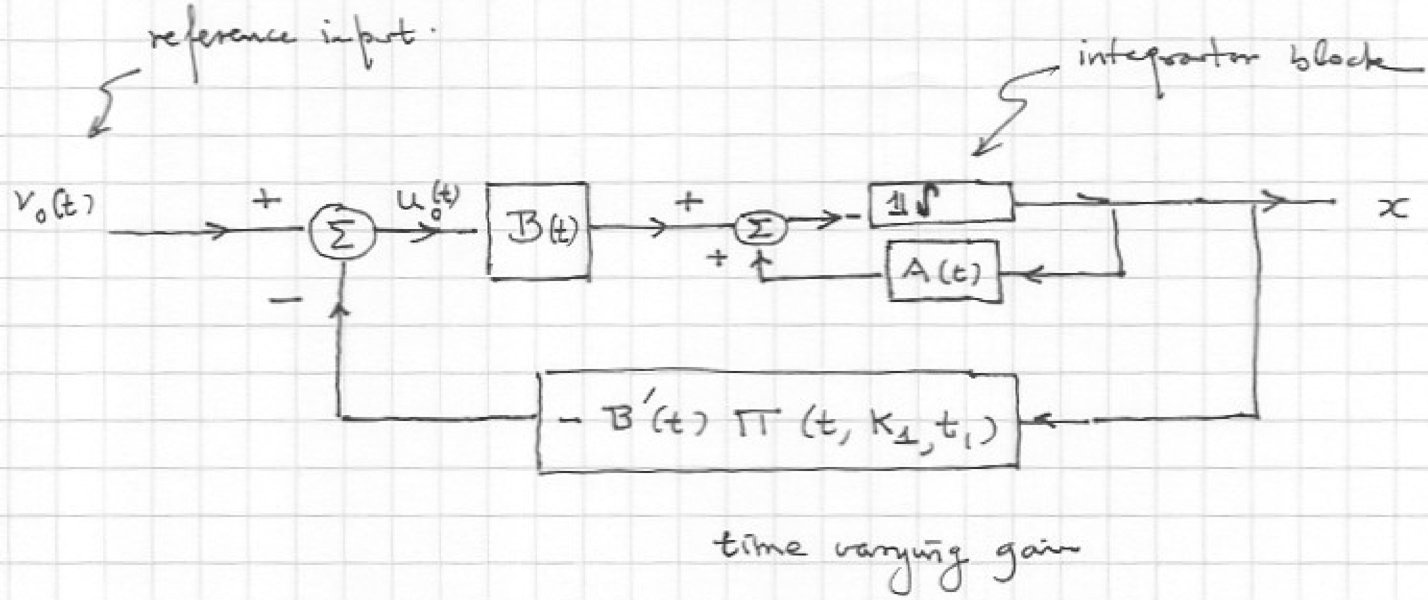
$\Phi_{A-BB'\Pi}$ is the corresponding

transition matrix.

Then

$$u_0(t) = \underset{\substack{\uparrow \\ \text{open loop term} \\ \text{(or reference input term)}}}{v_0(t)} - \underset{\substack{\uparrow \\ \text{feedback term}}}{B'(t)\,\Pi(t, K_1, t_1)\,x(t)}$$

This theorem shows the structure of solution to a fixed end-point optimal control problem. It takes the form of a sum of an open loop part and a feedback part.



Given (x_0, t_0) and (x_1, t_1) we can

(i) solve $\Pi(t, K_1, t_1)$

(ii) precompute $v_0(t)$

(iii) precompute the gain matrix $-B'(t)\,\Pi(t, K_1, t_1)$

step (ii) requires us to compute the Gramian

$$\tilde{W}(t_0, t_1) = \int_{t_0}^{t_1} \Phi_{A-BB'\Pi}(t_0, \sigma)\, B(\sigma) B'(\sigma)\, \Phi'_{A-BB'\Pi}(t_0, \sigma)\, d\sigma$$

which in turn requires us to compute the transition matrix $\Phi_{A-BB'\Pi}$ which satisfies

$$\frac{d}{dt} \Phi_{A-BB'\Pi}(t, t_0) = \left(A(t) - B(t) B'(t)\, \Pi(t, \cancel{0}^{K_1}, t_1)\right) \Phi_{A-BB'\Pi}(t, t_0)$$

Even if the original system is time invariant, this last step involves computing the transition matrix of a time varying system due to the time dependance of $\Pi(t, K_1, t_1)$. This is tedious!