Lecture 4 (Continued) ENEE 664. Spring 2004
PSK

Extrema.

Let X be a normed linear space. Let $f : \Omega \subset X \to \mathbb{R}$ be a functional defined on a subset Ω of X.

A point $x_0 \in \Omega$ is a relative extremum of f on Ω if there is an open ball $B_\varepsilon(x_0) = \{ \tilde{x} : \|\tilde{x} - x_0\| < \varepsilon \}$ such that

$$f(x_0) \le f(x) \quad \forall \quad x \in \Omega \cap B_\varepsilon(x_0)$$

i.e. x_0 is a relative minimum

OR

$$f(x_0) \ge f(x) \quad \forall \quad x \in \Omega \cap B_\varepsilon(x_0)$$

i.e. x_0 is a relative maximum.

If Ω is an open subset(†), a relative extremum is also referred to as a local extremum. If $\Omega = X$, relative extrema are just extrema.

Theorem 1 (necessary condition)

Let $f : X \to \mathbb{R}$ (or $f : \Omega \overset{open}{\subset} X \to \mathbb{R}$) be a ~~linear~~ functional which has a Gateaux differential on X (or Ω). A necessary condition for f to have an extremum at $x_0 \in X$ (or a local extremum at $x_0 \in \Omega \overset{open}{\subset} X$) is that $\delta f(x_0; h) = 0 \quad \forall \quad h \in X$.

(†) a set $\Omega \subset X$ is open, if given any $x \in \Omega$, there is an $\varepsilon > 0$ such that $x \in B_\varepsilon(x)$ and $B_\varepsilon(x) \subset \Omega$.

Proof: For $h \in X$, the function $\alpha \mapsto f(x_0 + \alpha h)$ achieves an extremum (or local extremum) at $\alpha = 0$. Then by calculus of one variable, $\left.\frac{d}{d\alpha} f(x_0 + \alpha h)\right|_{\alpha=0} = 0$. ▨

Theorem 2 Let $f: X \to \mathbb{R}$ be a functional on a vector space X. Suppose x_0 minimizes f on a convex set $\Omega \subset X$; and that f is Gateaux differentiable at x_0. Then,

$$\delta f(x_0; x - x_0) \geq 0 \qquad \forall\ x \in \Omega$$

Proof: Since Ω is convex and $x, x_0 \in \Omega$,

$$x_0 + \alpha(x - x_0) = \alpha x + (1-\alpha) x_0 \quad \in \Omega$$

for $\alpha \in [0, 1]$.

Then by calculus of one variable,

$$\left.\frac{d}{d\alpha} f(x_0 + \alpha(x - x_0))\right|_{\alpha=0} \geq 0$$

for a minimum at x_0, corresponding to $\alpha = 0$. ▨