ENEE 664, SPG 2004

Lecture Notes 9 (b)

These are additional notes on the mean value theorem in higher dimensions.

ENEE 661. Nonlinear Control Systems — PSK 02·21·00

Lecture 4 (part ii)

One of the basic results of single variable calculus is the classical mean value theorem (MVT)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on interval (a, b). There is c, $a < c < b$ such that the derivative

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \square$$

The adjoining picture gives us an idea of what's going on. The essential geometric idea is that at c (and c') the tangent to the graph of f is parallel to the line joining points $(a, f(a))$ and $(b, f(b))$.

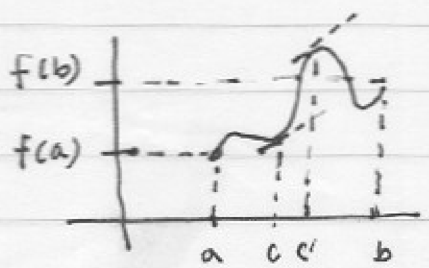

Figure 1. (MVT)

Let us see what happens in higher dimensions. Consider

$$f : [a, b] \to \mathbb{R}^2$$
$$x \mapsto f(x) = (y, z).$$

If the curve defined by f is of the "cork screw" variety then, there is no c, $a < c < b$ at which

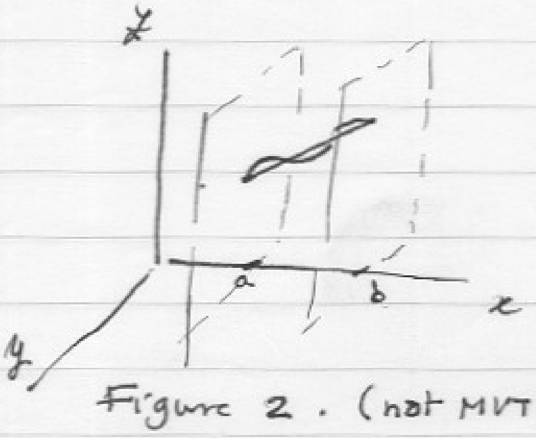

Figure 2. (not MVT)

PSK
02.21.00

the tangent to the curve is parallel to the line joining points $(a, f(a))$ and $(b, f(b))$ in the x-y-z space. The classical MVT does not hold. For a specific

Example: $f : [0, \frac{\pi}{2}] \to \mathbb{R}^2$

$$x \mapsto (\cos(x), \sin(x))$$

$$f(b) - f(a) = (-1, 1) \in \mathbb{R}^2$$
$$b - a = \frac{\pi}{2}$$

There does not excist $c \in [0, \frac{\pi}{2})$ such that $\frac{\pi}{2} \cdot (-\sin(c) \ \cos(c)) = (-1, 1)$ since it would require $\sin^2(c) + \cos^2(c) = \frac{8}{\pi} \neq 1$.

The correct form of mean value theorem in higher dimensions is actually an inequality. We need some preliminary results.

Lemma 1

Let $a < b$ $f : [a, b] \to V$ a normed linear space and $g : [a, b] \to \mathbb{R}$, f and g continuous on $[a, b]$ and differentiable on (a, b). Suppose,

$$\| f'(t) \| \leqq g'(t) \qquad a < t < b.$$

Then $\| f(b) - f(a) \| \leqq g(b) - g(a)$.

PSK
02-21-00

Proof (of lemma 1):

$$\|f(b)-f(a)\| = \left\| \int_a^b f'(\sigma)\,d\sigma \right\|$$

$$\leq \int_a^b \|f'(\sigma)\|\,d\sigma$$

$$\leq \int_a^b g'(\sigma)\,d\sigma$$

$$= g(b) - g(a) \qquad \square$$

Lemma 2

Same hypotheses as in Lemma 1, except that the condition on existence and inequality of derivatives holds for all $t \in [a,b]$ except for a countable set of points. Same conclusion as Lemma 1.

Proof: (essentially same argument as in Lemma 1 since the integrals are unaffected) $\square$

Corollary 3 Same hypotheses on f as in Lemma 1 and $g(t) = kt$, $k>0$ (thus $\|f'(t)\| \leq k \quad \forall\, t \in (a,b)$).

Then,

$$\|f(b)-f(a)\| \leq k\,(b-a). \qquad \square.$$

We need the definition of derivative for maps.

PSK
12.21.00

<u>Definition</u> Let E, F be normed linear spaces over $\mathbb{R}$. Let $U \overset{open}{\subset} \mathbb{R}$. Suppose
$$f : U \to F.$$
We say f is differentiable at $a \in U$ if there is a continuous <u>linear</u> map $L : E \to F$ such that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - Lh \|_F}{\| h \|_E} = 0.$$

Here h is such that $h + a \in U$.

Clearly, if L exists it is unique, and is given by
$$L(k) = \lim_{t \to 0} \frac{f(a+kt) - f(a)}{t}$$
$$= \frac{d}{dt} f(a+kt) \Big|_{t=0}$$

We call L the derivative of f at a and denote it by $(Df)_a$ and sometimes by $Df(a)$.

<u>Exercise:</u> (a) Chain Rule
$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

Composition of nonlinear maps | Composition of linear maps.

(b) If $E = \mathbb{R}^n$ & $F = \mathbb{R}^m$
$$(Df)(a) \cdot h = \left[\frac{\partial f^i}{\partial x_j} \right] h$$ Jacobian

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02-21-08

<u>Mean Value Theorem 4</u>

Let $f : U \overset{\text{open}}{\subset} E \to F$ be a map of normed linear spaces. Let $[a, b]$ denote the line segment

$$\{ (1-t)a + tb : 0 \le t \le 1 \},$$

with end-points $a, b \in U$, <u>contained</u> in U.

Then,

$$\| f(b) - f(a) \|$$

$$\le \sup_{0 \le t \le 1} \| Df[(1-t)a + tb] \| \cdot \|b - a\|$$

<u>Proof</u>: Simply restrict f to the line segment $[a, b]$ and then apply Corollary 3 above ▨

Another useful result from calculus is

<u>The Fundamental Theorem of Integral Calculus</u>.

Let X and Y be Banach spaces. Let $U \overset{\text{open}}{\subset} X$ and $f : U \to Y$ be a differentiable, everywhere in U, map or C^1 map. Suppose $x + ty \in U$ $\forall t \in [0, 1]$. Then

$$f(x+y) = f(x) + \int_0^1 Df(x+ty)y \, dt$$

PSK
02·21·08

<u>Proof</u>: The completeness / Banach property is used in the proper definition of integral with all attendant properties, as in 1 variable calculus. We take this for granted.

Then, set $g(t) = f(x+ty)$

$0 \le t \le 1.$

For $0 < t < 1$ by chain rule

$$g'(t) = Df(x+ty)\,y$$

Let $h(t) = f(x) + \int_0^t Df(x+sy)\,y\,ds$ $\quad 0 \le t \le 1$

Then $h'(t) = Df(x+ty)y$ $\quad 0 < t < 1$

Hence $g'(t) = h'(t)$

$\Rightarrow \quad g(t) = h(t) + \text{constant} \quad 0 < t < 1$

By continuity of g & h (they are integrals)

$g(t) = h(t) + \text{constant} \quad 0 \le t \le 1.$

But $g(0) = h(0) = f(x)$

So $g(1) = h(1) = f(x+y)$ ☐

<u>Lemma</u> (Differentiability and Lipschitz Condition).

Let $f: [a,b] \times D \to \mathbb{R}^n$ for domain $D \subset \mathbb{R}^n$, continuous in t and $\left(\frac{\partial f}{\partial x}\right)$ exists and is continuous on $[a,b] \times D$. Then f is <u>locally</u>

PSK
02-21-10

<u>Lipschitz</u> on $[a, b] \times \mathcal{D}$.

<u>Proof:</u> For $x_0 \in \mathcal{D}$, let $r > 0$ be such that

$$\mathcal{D}_0 \equiv \{x \,/\, \|x - x_0\| \le r\} \subset \mathcal{D}.$$

$\mathcal{D}_0$ is closed and bounded. $\mathcal{D}_0$ is convex, since, for $x_1, x_2 \in \mathcal{D}_0$ and $0 \le \alpha \le 1$,

$$\begin{aligned}
\|\alpha x_1 + (1-\alpha)x_2 - x_0\| &= \|\alpha x_1 + (1-\alpha)x_2 - \alpha x_0 - (1-\alpha)x_0\| \\
&= \|\alpha(x_1 - x_0) + (1-\alpha)(x_2 - x_0)\| \\
&\le \alpha \|x_1 - x_0\| + (1-\alpha)\|x_2 - x_0\| \\
&\le \alpha r + (1-\alpha) r \\
&= r.
\end{aligned}$$

Then, by the Mean Value Theorem, $\forall\, x, y \in \mathcal{D}_0$

$$\begin{aligned}
&\|f(t, y) - f(t, x)\| \\
&\quad \le \sup_{0 \le s \le 1} \|Df((1-s)x + sy\| \cdot \|y - x\| \\
&\quad \le \sup_{t \in [a,b]} \sup_{0 \le s \le 1} \|Df((1-s)x + sy)\| \cdot \|y - x\| \\
&\quad = L \cdot \|y - x\|,
\end{aligned}$$

where we used continuity wrt both t & x of Df in the sup norms.

addendum: (02/26/02)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 (i.e. continuously differentiable) at each point x of an open set $S \subset \mathbb{R}^n$.

Suppose $x^*, y^* \in S$ are such that the line segment $L(x^*, y^*)$ joining x^* and $y^* \subset S$. Then there exists a point $z \in L(x^*, y^*)$ such that

$$f(y^*) - f(x^*) = \left(\frac{\partial f}{\partial x}\right)_{x=z} (y - x)$$

Proof Let $g(t) = f((1-t)x^* + ty^*)$

Then $g(0) = f(x^*)$

$g(1) = f(y^*)$.

$$g'(t) = \left(\frac{\partial f}{\partial x}\right)_{x=(1-t)x^*+ty^*} \cdot \frac{d}{dt}\left((1-t)x^* + ty^*\right)$$

$$= \left(\frac{\partial f}{\partial x}\right)_{x=(1-t)x^*+ty^*} \cdot (y^* - x^*)$$

By the scalar mean value theorem

$$\left(g(1) - g(0)\right) = g'(t)\Big|_{t=t^*} \cdot (1-0)$$

which is the same as saying

$$f(y^*) - f(x^*) = \left(\frac{\partial f}{\partial x}\right)_{x=z} \cdot (y^* - x^*)$$

where $z = (1-t^*)x^* + t^*y^* \in L(x, y)$ ▨