that $\lim_{\alpha \to 0} \dfrac{T(x+\alpha h) - T(x)}{\alpha} = DT(x;h)$

i.e. $\delta T(x;h) = DT(x;h)$ ▨

<u>Proposition 5</u> (continuity from differentiability)

If $T: U \subset X \to Y$ is Fréchet differentiable at x then T is continuous at x, (Here $x \in U$).

<u>Proof</u> Given $\epsilon > 0$, there is a ball centered at x of radius ε: $B_\varepsilon(x) = \{\tilde{x} \in U : \|\tilde{x} - x\| < \varepsilon\} \subset U$ provided ε is sufficiently small (since U is open).

For $x + h \in B_\varepsilon(x)$,

$$\|T(x+h) - T(x) - DT(x;h)\| \leq \varepsilon \|h\|.$$

Thus

$$\begin{aligned} \|T(x+h) - T(x)\| &\leq \varepsilon \|h\| + \|DT(x;h)\| \\ &\leq \varepsilon \|h\| + \|DT(x;\cdot)\| \, \|h\| \\ &\leq \left(\varepsilon + \|DT(x;\cdot)\|\right) \|h\| \\ &= M \|h\| \end{aligned}$$ ▨

So pick $\delta = \varepsilon / M$ to get continuity of T

<u>Example</u> Let $f: \mathbb{R}^n \to \mathbb{R}$ have continuous first partial derivatives at $x_0 \in \mathbb{R}^n$. Then the differential $\delta f(x_0;h) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{|x=x_0} \cdot h_i$ is <u>the</u> Fréchet differential.