ENEE 664 Optimal Control - Introduction 01/27/2016

1* Consider Figure 1 below of the graph of a function f of a real variable x.

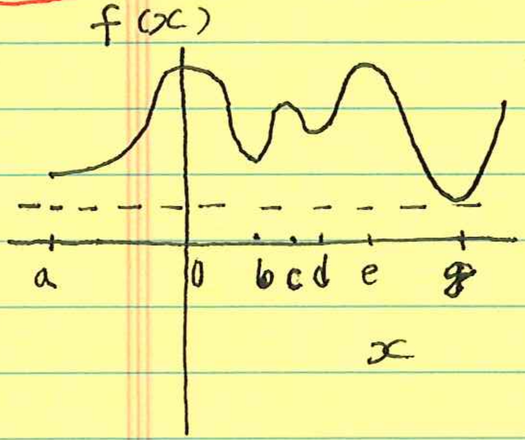


Figure 1

At $x = 0, b, c, d, e, g$ the derivative

$$f'(x)$$

vanishes (tangent horizontal)

When asked to solve the problem

$$\text{Minimize } f(x)$$
$$-\infty < x < \infty$$

your candidate point would be $x = g$ but you cannot be sure since you lack global information as to the behavior of the function farther out on the real axis. When asked to solve the constrained problem

$$\text{Minimize } f(x)$$
$$x_1 < x < x_2$$

for x_1 and x_2 in the interval (a, g) your candidate points would be $x = b$ and d, respectively, if $x_1 < b < c$, respectively $c < x_1 < e$.

But if $0 < x_1 < x_2 < b$ there is no solution to the constrained problem! Closer you get to 'b' smaller f is, but you do not get to 'b'

More simply, if $f(x) = e^{-x^2}$ there is no x^* that solves the problem $\displaystyle\min_{-\infty<x<\infty} e^{-x^2}$. But for real numbers a and b, the problem $\displaystyle\min_{a \le x \le b} e^{-x^2}$ has a solution – one of the end points of the interval $[a, b]$, or both end points if $a = -b$.

Remark Minimization problems may or may not have a solution depending on the constraint.

NEWTON

Remark Solving $f'(x) = 0$ for candidates x^* is key

2 * Consider Figure 2 below showing a rocket car on a track (real axis) with bi-directional thrusters – thus you can push left or right

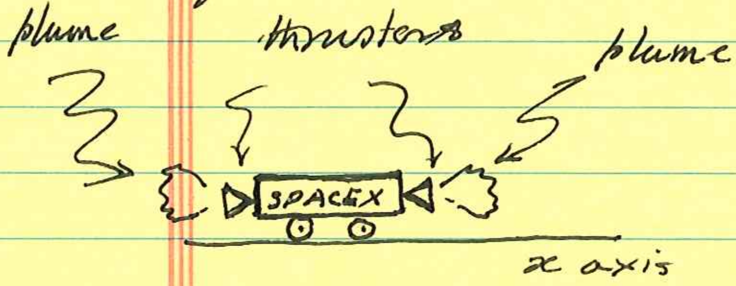


Figure 2

Modeling this system as a point mass we write

$$\boxed{\ddot{x} = u}$$ NEWTON

where x is the location on the track and u = net thrust (per unit mass) = CONTROL integrator

We can represent this as a <u>cascade</u> of $\int$

$$u \longrightarrow \boxed{\int} \xrightarrow{x_2} \boxed{\int} \longrightarrow x = x_1$$

we write

$$(*) \qquad \begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= u \end{aligned}$$

a linear system with 2 dimensional state space

A typical "mission" may be

$$(\mathrm{I}) \qquad \min_{u(\cdot)} \int_0^T u^2(t)\, dt$$

KALMAN

subject to (*) above

$x_1(0) = x_2(0) = 0$

$x_1(T) = a \qquad x_2(T) = 0$

and a, T specified

This may be characterized as "energy optimal" rest-to-rest transfer. It is clear that there are many such rest-to-rest transfers out of which there may be one or many possible thrust profiles (with restrictions, such as continuity, differentiability, etc on $u(\cdot)$) that optimise.

(II) $$\min_{u(\cdot)} \int_0^T 1 \cdot dt = T$$

subject to (*) above

$x_1(0) = x_2(0) = 0$

$x_1(T) = a, \quad x_2(T) = 0$

a specified

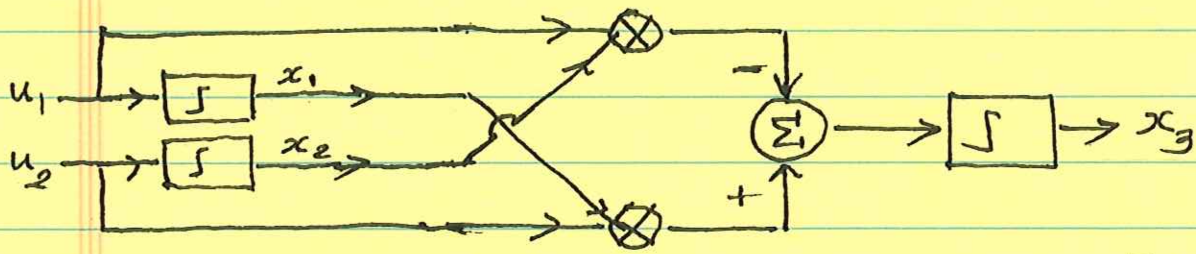
You are asked to minimize time to transfer, but this does not make sense (arbitrarily large magnitude of $u(t)$ can accomplish arbitrarily quick transfer) - So you need some limits imposed e.g. $|u(t)| \leq M$

This time optimal control problem leads to "hitting the rails" or "bang-bang" solution $u(t) = +M$ followed by $u(t) = -M$. or vice versa.

Remark Both missions are expressed by integral cost functionals ~ sort of a standard form as we shall see

Remark The example generalizes to systems with a cascade of more integrators (higher dimensional linear systems)

$\boxed{3 *}$ Another, very important, class of optimal control problems combines integrators and multipliers (and hence nonlinear). See Figure 3 below displaying what is known as the nonholonomic integrator (BROCKETT)



Σ : adder

Figure 3

$\otimes$: multiplier

This system of 3 integrators, 2 multipliers and 1 adder, is expressed by the differential equation

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2 \qquad \text{(BROCKETT)}$$

$$\dot{x}_3 = x_1 u_2 - x_2 u_1$$

We notice that the last equation, written as

(N) $\dot{x}_3 + x_2 \dot{x}_1 - x_1 \dot{x}_2 \equiv 0$ (CARATHEODORY)

is a constraint on curves in 3 dimensional space which cannot be expressed in the form of (surface equation)

(H) $\qquad \phi\,(x_1(t), x_2(t), x_3(t)) \equiv \text{constant}$

for any differentaiable ϕ. If it were possible to do so, differentiation yields

$$\frac{\partial \phi}{\partial x_1}\dot{x}_1 + \frac{\partial \phi}{\partial x_2}\dot{x}_2 + \frac{\partial \phi}{\partial x_3}\dot{x}_3 \equiv 0$$

which, to agree with (N) above would mean

$$\frac{\partial \phi}{\partial x_1} = x_2 \; ; \; \frac{\partial \phi}{\partial x_2} = -x_1$$

$$\frac{\partial \phi}{\partial x_3} = 1$$

implying,

$$\frac{\partial^2 \phi}{\partial x_2 \partial x_1} = 1 \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -1$$

a contradiction!

Constraints of the form (H) are called holonomic constraints, and since (N) does <u>not</u> arise from <u>any</u> (H) we say the system of Figure 3 is a nonholonomic integrator.

Remark The nonholonomic integrator is a rectifier for the following reason. Suppose u_1 and u_2 are zero mean periodic functions of common period T then

$$\Delta x_3 = x_3(T) - x_3(0)$$

$$= \int_0^T \left(x_1(t)\, \dot{x}_2(t) - x_2(t)\, \dot{x}_1(t) \right) dt$$

$$= \oint_\gamma \left(x_1\, dx_2 - x_2\, dx_1 \right)$$

a loop integral where γ is the loop traced in the (x_1, x_2) plane. By Green / Stokes theorem

$$\Delta x_3 = \iint_S 2\, dx_1 dx_2$$

$$= 2 \text{ area of region } S$$

$$\text{enclosed by } \gamma$$

Thus in each period x_3 increases an amount equal to 2·area – conversion of AC inputs into DC. Hence a rectifier. Such rectification is essential in biological locomotion at all scales

Consider the optimal control problem

$$\text{Min} \int_0^T (u_1^2(t) + u_2^2(t))\,dt$$

s.t.

$x_1(0) = x_2(0) = x_1(T) = x_2(T) = 0$

$\Delta x_3 = 2A$

x_1, x_2, x_3 obey BROCKETT

We will see that replacing the cost functional by $\int_0^T \sqrt{u_1^2 + u_2^2}$ does not change the answer, but this is the problem of minimizing the perimeter of a closed loop with prescribed area — classically a circle. Optimal control problems of this type are rich.

Existence of any control at all that solves the problem of transfer of x_1, x_2, x_3 from anyplace to any other by the nonholonomic integrator is a controllability problem answered in the affirmative by using geometric principles

$\boxed{4 *}$

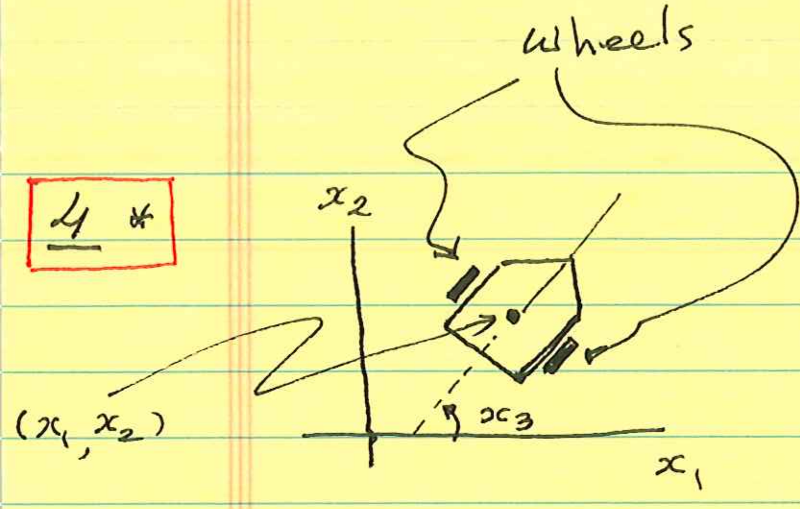


Figure 4

For a robot (see top view in Figure 4) equipped with two independently controlled wheels and a castor (not shown), at any moment in time, location (x_1, x_2) and orientation x_3 determine the configuration. You may also view this as a unicycle (wheelbase shrinks to zero). You control heading speed v and steering rate u with associated equation

$$\dot{x}_1 = v \cos(x_3)$$

$$\dot{x}_2 = v \sin(x_3) \qquad \text{(EUCLID/LIE)}$$

$$\dot{x}_3 = u$$

Typical Missions (subject to EUCLID/LIE)

(I) $\text{Min} \int_0^T (u^2(t) + v^2(t))\,dt$

$x_1(0) = x_2(0) = x_3(0) = 0$

$x_1(T) = x_2(T) = 1; \quad x_3(T) = \pi/2$

$u(\cdot), v(\cdot)$ continuous

II $\quad \text{Min} \int_0^T 1 \cdot dt \qquad T$ not specified

$|u(t)| \le 1$

$x_1(0) = x_2(0) = x_3(0) = 0$

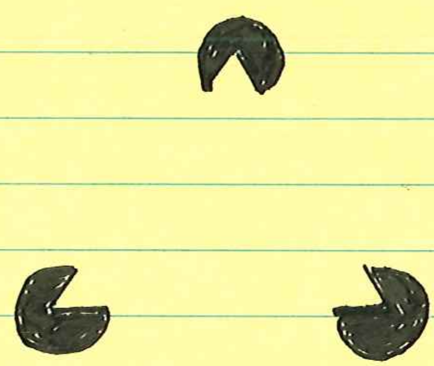
$x_1(T) = x_2(T) = 1 \,; \quad x_3(T) = 3\pi/2$

$u(\cdot), v(\cdot)$ continuous

These are classic motion planning problems of robotics.

5*

Consider Figure 5. Do you see a triangle pop out even though there are no real edges? This virtual triangle (called Kanisza triangle) is the perception of your visual system, a reconstruction by



your brain thanks to computation / completion of curves subject to boundary conditions, here specified by the 'pacman' disks! Such illusions suggest that the brain is solving optimization problems to 'make sense' of data.