I attach two problem sets from H. Khalil on Nonlinear Systems. There are two attachments.
(a) on o.d.e. basics and (problems (1, 3, 4, 6, 8, 17, 23)
(b) on Lyapunov theory. (1, 3, 5, 6, 7, 15, 17, 21, 22)

Due date - Oct 2

Students who have not taken nonlinear control theory (equivalent to ENEE 661) here or elsewhere can ask me for help, including special group discussions.

For those who have taken that material, I ask you to do these (honor system)
without looking up any solutions and simply show your best understanding of the material from that class. This would be just a warm up for you.

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3.5 Exercises

3.1 For each of the functions $f(x)$ given next, find whether f is (a) continuously differentiable; (b) locally Lipschitz; (c) continuous; (d) globally Lipschitz.

(1) $f(x) = x^2 + |x|$.
(2) $f(x) = x + \text{sgn}(x)$.
(3) $f(x) = \sin(x)\,\text{sgn}(x)$.
(4) $f(x) = -x + a\sin(x)$.
(5) $f(x) = -x + 2|x|$.
(6) $f(x) = \tan(x)$.

(7) $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) - \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$.

(8) $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$.

3.2 Let $D_r = \{x \in R^n \mid \|x\| < r\}$. For each of the following systems, represented as $\dot{x} = f(t, x)$, find whether (a) f is locally Lipschitz in x on D_r, for sufficiently small r; (b) f is locally Lipschitz in x on D_r, for any finite $r > 0$; (c) f is globally Lipschitz in x:

(1) The pendulum equation with friction and constant input torque (Section 1.2.1).

(2) The tunnel-diode circuit (Example 2.1).

(3) The mass–spring equation with linear spring, linear viscous damping, Coulomb friction, and zero external force (Section 1.2.3).

(4) The Van der Pol oscillator (Example 2.6).

(5) The closed-loop equation of a third-order adaptive control system (Section 1.2.5).

(6) The system $\dot{x} = Ax - B\psi(Cx)$, where A, B, and C are $n \times n$, $n \times 1$, and $1 \times n$ matrices, respectively, and $\psi(\cdot)$ is the dead-zone nonlinearity of Figure 1.10(c).

3.3 Show that if $f_1 : R \to R$ and $f_2 : R \to R$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$ and $f_2 \circ f_1$ are locally Lipschitz.

3.4 Let $f : R^n \to R^n$ be defined by

$$f(x) = \begin{cases} \frac{1}{\|Kx\|}Kx, & \text{if } g(x)\|Kx\| \ge \mu > 0 \\ \frac{g(x)}{\mu}Kx, & \text{if } g(x)\|Kx\| < \mu \end{cases}$$

where $g : R^n \to R$ is locally Lipschitz and nonnegative, and K is a constant matrix. Show that $f(x)$ is Lipschitz on any compact subset of R^n.

3.5 Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two different p-norms on R^n. Show that $f : R^n \to R^m$ is Lipschitz in $\|\cdot\|_\alpha$ if and only if it is Lipschitz in $\|\cdot\|_\beta$.

3.6 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \leq k_1 + k_2\|x\|, \quad \forall\ (t,x) \in [t_0,\infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \leq \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}$$

for all $t \geq t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

3.7 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{1}{1+g^T(x)g(x)} g(x)$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

3.8 Show that the state equation

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b \end{aligned}$$

has a unique solution defined for all $t \geq 0$.

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

3.10 Derive the sensitivity equations for the tunnel-diode circuit of Example 2.1 as L and C vary from their nominal values.

3.11 Derive the sensitivity equations for the Van der Pol oscillator of Example 2.6 as ε varies from its nominal value. Use the state equation in the x-coordinates.

3.12 Repeat the previous exercise by using the state equation in the z-coordinates.

3.13 Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

3.14 Consider the system

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2) \\ \dot{x}_2 &= -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2)\end{aligned}$$

where λ and τ are positive constants.

(a) Derive the sensitivity equations as λ and τ vary from their nominal values λ_0 and τ_0.

(b) Show that $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \le -\frac{1}{\tau}r + 2\sqrt{2}$$

(c) Using the comparison lemma, show that the solution of the state equation satisfies the inequality

$$\|x(t)\|_2 \le e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}\tau(1 - e^{-t/\tau})$$

3.15 Using the comparison lemma, show that the solution of the state equation

$$\dot{x}_1 = -x_1 + \frac{2x_2}{1 + x_2^2}, \qquad \dot{x}_2 = -x_2 + \frac{2x_1}{1 + x_1^2}$$

satisfies the inequality

$$\|x(t)\|_2 \le e^{-t}\|x(0)\|_2 + \sqrt{2}\,(1 - e^{-t})$$

3.16 Using the comparison lemma, find an upper bound on the solution of the scalar equation

$$\dot{x} = -x + \frac{\sin t}{1 + x^2}, \quad x(0) = 2$$

3.17 Consider the initial-value problem (3.1) and let $D \subset R^n$ be a domain that contains $x = 0$. Suppose $x(t)$, the solution of (3.1), belongs to D for all $t \ge t_0$ and $\|f(t,x)\|_2 \le L\|x\|_2$ on $[t_0, \infty) \times D$. Show that

(a)

$$\left| \frac{d}{dt}\left[x^T(t)x(t)\right] \right| \le 2L\|x(t)\|_2^2$$

(b)

$$\|x_0\|_2 \exp[-L(t - t_0)] \le \|x(t)\|_2 \le \|x_0\|_2 \exp[L(t - t_0)]$$

3.18 Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \le k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]\ d\tau$$

where k_1, k_2, and k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall–Bellman inequality, show that

$$y(t) \le k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

3.19 Let $f : R^n \to R^n$ be locally Lipschitz in a domain $D \subset R^n$. Let $S \subset D$ be a compact set. Show that there is a positive constant L such that for all $x, y \in S$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

Hint: The set S can be covered by a finite number of neighborhoods; that is,

$$S \subset N(a_1, r_1) \cup N(a_2, r_2) \cup \cdots \cup N(a_k, r_k)$$

Consider the following two cases separately:

- $x, y \in S \cap N(a_i, r_i)$ for some i.
- $x, y \notin S \cap N(a_i, r_i)$ for any i; in this case, $\|x - y\| \ge \min_i r_i$.

In the second case, use the fact that $f(x)$ is uniformly bounded on S.

3.20 Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

3.21 For any $x \in R^n - \{0\}$ and any $p \in [1, \infty)$, define $y \in R^n$ by

$$y_i = \frac{x_i^{p-1}}{\|x\|_p^{p-1}}\ \mathrm{sign}(x_i^p)$$

Show that $y^T x = \|x\|_p$ and $\|y\|_q = 1$, where $q \in (1, \infty]$ is determined from $1/p + 1/q = 1$. For $p = \infty$, find a vector y such that $y^T x = \|x\|_\infty$ and $\|y\|_1 = 1$.

3.22 Prove Lemma 3.3.

3.23 Let $f(x)$ be a continuously differentiable function that maps a convex domain $D \subset R^n$ into R^n. Suppose D contains the origin $x = 0$ and $f(0) = 0$. Show that

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\ d\sigma\ x, \quad \forall\ x \in D$$

Hint: Set $g(\sigma) = f(\sigma x)$ for $0 \le \sigma \le 1$ and use the fact that $g(1) - g(0) = \int_0^1 g'(\sigma)\ d\sigma$.

3.24 Let $V : R \times R^n \to R$ be continuously differentiable. Suppose that $V(t, 0) = 0$ for all $t \geq 0$ and

$$V(t,x) \geq c_1\|x\|^2; \quad \left\|\frac{\partial V}{\partial x}(t,x)\right\| \leq c_4\|x\|, \quad \forall\ (t,x) \in [0,\infty) \times D$$

where c_1 and c_4 are positive constants and $D \subset R^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \leq \frac{1}{2}c_4\|x\|^2$ for all $x \in D$.
Hint: Use the representation $V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x)\ d\sigma\ x$.

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \leq c_4$.

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$|W(t,x_2) - W(t,x_1)| \leq \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\|, \quad \forall\ t \geq 0,\ \forall\ x_1, x_2 \in D$$

3.25 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0, t_1] \times D$, for some domain $D \subset R^n$. Let W be a compact subset of D. Let $x(t)$ be the solution of $\dot{x} = f(t,x)$ starting at $x(t_0) = x_0 \in W$. Suppose that $x(t)$ is defined and $x(t) \in W$ for all $t \in [t_0, T)$, $T < t_1$.

(a) Show that $x(t)$ is uniformly continuous on $[t_0, T)$.

(b) Show that $x(T)$ is defined and belongs to W and $x(t)$ is a solution on $[t_0, T]$.

(c) Show that there is $\delta > 0$ such that the solution can be extended to $[t_0, T + \delta]$.

3.26 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0, t_1] \times D$, for some domain $D \subset R^n$. Let $y(t)$ be a solution of (3.1) on a maximal open interval $[t_0, T) \subset [t_0, t_1]$ with $T < \infty$. Let W be any compact subset of D. Show that there is some $t \in [t_0, T)$ with $y(t) \notin W$.
Hint: Use the previous exercise.

3.27 ([43]) Let $x_1 : R \to R^n$ and $x_2 : R \to R^n$ be differentiable functions such that

$$\|x_1(a) - x_2(a)\| \leq \gamma, \quad \|\dot{x}_i(t) - f((t, x_i(t))\| \leq \mu_i, \text{ for } i = 1, 2$$

for $a \leq t \leq b$. If f satisfies the Lipschitz condition (3.2), show that

$$\|x_1(t) - x_2(t)\| \leq \gamma e^{L(t-a)} + (\mu_1 + \mu_2)\left[\frac{e^{L(t-a)} - 1}{L}\right], \quad \text{for } a \leq t \leq b$$

3.28 Show, under the assumptions of Theorem 3.5, that the solution of (3.1) depends continuously on the initial time t_0.

3.29 Let $f(t,x)$ and its partial derivatives with respect to x be continuous in (t,x) for all $(t,x) \in [t_0, t_1] \times R^n$. Let $x(t,\eta)$ be the solution of (3.1) that starts at $x(t_0) = \eta$ and suppose $x(t,\eta)$ is defined on $[t_0, t_1]$. Show that $x(t,\eta)$ is continuously differentiable with respect to η and find the variational equation satisfied by $[\partial x/\partial \eta]$. Hint: Put $y = x - \eta$ to transform (3.1) into

$$\dot{y} = f(t, y + \eta), \quad y(t_0) = 0$$

with η as a parameter.

3.30 Let $f(t,x)$ and its partial derivative with respect to x be continuous in (t,x) for all $(t,x) \in R \times R^n$. Let $x(t,a,\eta)$ be the solution of (3.1) that starts at $x(a) = \eta$ and suppose that $x(t,a,\eta)$ is defined on $[a, t_1]$. Show that $x(t,a,\eta)$ is continuously differentiable with respect to a and η and let $x_a(t)$ and $x_\eta(t)$ denote $[\partial x/\partial a]$ and $[\partial x/\partial \eta]$, respectively. Show that $x_a(t)$ and $x_\eta(t)$ satisfy the identity

$$x_a(t) + x_\eta(t) f(a,\eta) \equiv 0, \quad \forall\ t \in [a, t_1]$$

3.31 (**[43]**) Let $f : R \times R \to R$ be a continuous function. Suppose that $f(t,x)$ is locally Lipschitz and nondecreasing in x for each fixed value of t. Let $x(t)$ be a solution of $\dot{x} = f(t,x)$ on an interval $[a,b]$. If a continuous function $y(t)$ satisfies the integral inequality

$$y(t) \le x(a) + \int_a^t f(s, y(s))\ ds$$

for $a \le t \le b$, show that $y(t) \le x(t)$ throughout this interval.

4.10 Exercises

4.1 Consider a second-order autonomous system. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable:

(1) stable node	**(2)** unstable node	**(3)** stable focus
(4) unstable focus	**(5)** center	**(6)** saddle

Justify your answer using phase portraits.

4.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

4.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable:

$$\begin{array}{lllllll}
(1) & \dot{x}_1 & = & -x_1 + x_1 x_2, & \dot{x}_2 & = & -x_2 \\
(2) & \dot{x}_1 & = & -x_2 - x_1(1 - x_1^2 - x_2^2), & \dot{x}_2 & = & x_1 - x_2(1 - x_1^2 - x_2^2) \\
(3) & \dot{x}_1 & = & x_2(1 - x_1^2), & \dot{x}_2 & = & -(x_1 + x_2)(1 - x_1^2) \\
(4) & \dot{x}_1 & = & -x_1 - x_2, & \dot{x}_2 & = & 2x_1 - x_2^3
\end{array}$$

Investigate whether the origin is globally asymptotically stable.

4.4 (**[151]**) Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$, where k_1 to k_3 are positive constants. Show that the origin of the closed-loop system is globally asymptotically stable.

4.5 Let $g(x)$ be a map from R^n into R^n. Show that $g(x)$ is the gradient vector of a scalar function $V : R^n \to R$ if and only if

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\, i, j = 1, 2, \ldots, n$$

4.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \in R$. Using th variable gradient method, find a Lyapunov function that shows that the origin i globally asymptotically stable.

4.7 Consider the system $\dot{x} = -Q\phi(x)$, where Q is a symmetric positive definit matrix and $\phi(x)$ is a continuously differentiable function for which the ith compc nent ϕ_i depends only on x_i, that is, $\phi_i(x) = \phi_i(x_i)$. Assume that $\phi_i(0) = 0$ an $y\phi_i(y) > 0$ in some neighborhood of $y = 0$, for all $1 \le i \le n$.

(a) Using the variable gradient method, find a Lyapunov function that shows thε the origin is asymptotically stable.

(b) Under what conditions will it be globally asymptotically stable?

(c) Apply to the case

$$n = 2, \ \phi_1(x_1) = x_1 - x_1^2, \ \phi_2(x_2) = x_2 + x_2^3, \ Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4.8 ([**72**]) Consider the second-order system

$$\dot{x}_1 = \frac{-6x_1}{u^2} + 2x_2, \qquad \dot{x}_2 = \frac{-2(x_1 + x_2)}{u^2}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vect field on the boundary of this hyperbola, that trajectories to the right of t branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, a compare with the slope of the tangents to the hyperbola.

4.9 In checking radial unboundedness of a positive definite function $V(x)$, it n appear that it is sufficient to examine $V(x)$ as $\|x\| \to \infty$ along the principal ax This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \le -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x \, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \le -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem:
For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem:
For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \ge 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$(1) \qquad \dot{x}_1 = x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$(2) \qquad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \le x_1 \le 1\} \cap \{x_2 \ge x_1^3\} \cap \{x_2 \le x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \ge 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y)\, dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\ dy + x_2^2/2 + \int_0^{x_3} h_2(y)\ dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

4.17 (**[77]**) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (1/2)\left[x_2 + \int_0^{x_1} h(y)\ dy\right]^2 + \int_0^{x_1} g(y)\ dy$.

4.18 The mass–spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

4.19 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated roots at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin $(q = 0,\ \dot{q} = 0)$ is stable.

(b) With $u = -K_d\dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d\dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point $(q = q^*,\ \dot{q} = 0)$ is an asymptotically stable equilibrium point.

4.20 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

4.21 ([81]) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \geq 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

4.22 Consider the Lyapunov equation $PA + A^TP = -C^TC$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0\ \forall\ t$ if and only if $x = 0$.

4.23 Consider the linear system $\dot{x} = (A - BR^{-1}B^TP)x$, where $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^TPx$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

(1) $Q > 0$.

(2) $Q = C^TC$ and (A, C) is observable; see the hint of Exercise 4.22.

4.24 Consider the system[36]

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

[36] This is a closed-loop system under optimal stabilizing control. See [172].

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x} f(x) + q(x) - \tfrac{1}{4}\frac{\partial V}{\partial x} G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a nonsingular matrix, and k is a positive constant. Using $V(x)$ as a Lyapunov function candidate, show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

When will the origin be globally asymptotically stable?

4.25 Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At}BB^Te^{-A^Tt}\, dt$ for some $\tau > 0$. Show that W is positive definite and let $K = B^TW^{-1}$. Use $V(x) = x^TW^{-1}x$ as a Lyapunov function candidate for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz.

4.26 Let $\dot{x} = f(x)$, where $f : R^n \rightarrow R^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : R^n \rightarrow R^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable or unstable) if and only if $z = 0$ is stable (asymptotically stable or unstable).

4.27 Consider the system

$$\dot{x}_1 = -x_2x_3 + 1, \qquad \dot{x}_2 = x_1x_3 - x_2, \qquad \dot{x}_3 = x_3^2(1 - x_3)$$

(a) Show that the system has a unique equilibrium point.

(b) Using linearization, show that the equilibrium point asymptotically stable. Is it globally asymptotically stable?

4.28 Consider the system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show, by using linearization, that $x = 0$ is asymptotically stable.

(c) Show that $\Gamma = \{x \in R^2 \mid x_1 x_2 \geq 2\}$ is a positively invariant set.

(d) Is $x = 0$ globally asymptotically stable?

4.29 Consider the system

$$\dot{x}_1 = x_1 - x_1^3 + x_2, \qquad \dot{x}_2 = 3x_1 - x_2$$

(a) Find all equilibrium point of the system.

(b) Using linearization, study the stability of each equilibrium point.

(c) Using quadratic Lyapunov functions, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimate as large as possible.

(d) Construct the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

4.30 Repeat the previous exercise for the system

$$\dot{x}_1 = -\tfrac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2, \qquad \dot{x}_2 = x_1 - \tfrac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)$$

4.31 For each of the systems of Exercise 4.3, use linearization to show that the origin is asymptotically stable.

4.32 For each for the following systems, investigate whether the origin is stable, asymptotically stable, or unstable:

$$(1)\quad \begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 \\ \dot{x}_2 &= -x_2 + x_3^2 \\ \dot{x}_3 &= x_3 - x_1^2 \end{aligned} \qquad (2)\quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_3 + x_1[-2x_3 - \text{sat}(y)]^2 \\ \dot{x}_3 &= -2x_3 - \text{sat}(y) \\ &\text{where } y = -2x_1 - 5x_2 + 2x_3 \end{aligned}$$

$$(3)\quad \begin{aligned} \dot{x}_1 &= -2x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_3 \end{aligned} \qquad (4)\quad \begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_1 x_3 \\ \dot{x}_3 &= (x_1 + 1)x_2 \end{aligned}$$

4.33 Consider the second-order system $\dot{x} = f(x)$, where $f(0) = 0$ and $f(x)$ is twice continuously differentiable in some neighborhood of the origin. Suppose $[\partial f/\partial x](0) = -B$, where B be Hurwitz. Let P be the positive definite solution of the Lyapunov equation $PB + B^T P = -I$ and take $V(x) = x^T P x$. Show that there exists $c^* > 0$ such that, for every $0 < c < c^*$, the surface $V(x) = c$ is closed and $[\partial V/\partial x] f(x) > 0$ for all $x \in \{V(x) = c\}$.

4.34 Prove Lemma 4.2.

4.35 Let α be a class $\mathcal{K}$ function on $[0, a)$. Show that

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2), \quad \forall\, r_1, r_2 \in [0, a)$$

4.36 Is the origin of the scalar system $\dot{x} = -x/(t+1)$, $t \geq 0$, uniformly asymptotically stable?

4.37 For each of the following linear systems, use a quadratic Lyapunov function to show that the origin is exponentially stable:

$$(1)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x,\ |\alpha(t)| \leq 1 \qquad (2)\ \dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

$$(3)\ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x,\ \alpha(t) \geq 2 \qquad (4)\ \dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

In all cases, $\alpha(t)$ is continuous and bounded for all $t \geq 0$.

4.38 (**[95]**) An *RLC* circuit with time-varying elements is represented by

$$\dot{x}_1 = \frac{1}{L(t)} x_2, \qquad \dot{x}_2 = -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2$$

Suppose that $L(t)$, $C(t)$, and $R(t)$ are continuously differentiable and satisfy the inequalities $k_1 \leq L(t) \leq k_2$, $k_3 \leq C(t) \leq k_4$, and $k_5 \leq R(t) \leq k_6$ for all $t \geq 0$, where k_1, k_3, and k_5 are positive. Consider a Lyapunov function candidate

$$V(t, x) = \left[R(t) + \frac{2L(t)}{R(t)C(t)}\right] x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

(a) Show that $V(t, x)$ is positive definite and decrescent.

(b) Find conditions on $\dot{L}(t)$, $\dot{C}(t)$, and $\dot{R}(t)$ that will ensure exponential stability of the origin.

4.39 (**[154]**) A pendulum with time-varying friction is represented by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - g(t)x_2$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty \quad \text{and} \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t, x) = \tfrac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

4.40 (Floquet theory) Consider the linear system $\dot{x} = A(t)x$, where $A(t) = A(t+T)$.[37] Let $\Phi(\cdot,\cdot)$ be the state transition matrix. Define a constant matrix B via the equation $\exp(BT) = \Phi(T,0)$, and let $P(t) = \exp(Bt)\Phi(0,t)$. Show that

(a) $P(t+T) = P(t)$.

(b) $\Phi(t,\tau) = P^{-1}(t)\exp[(t-\tau)B]P(\tau)$.

(c) the origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

4.41 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

4.42 Consider the system

$$\dot{x} = -a[I_n + S(x) + xx^T]x$$

where a is a positive constant, I_n is the $n \times n$ identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Show that the origin is globally exponentially stable.

4.43 Consider the system $\dot{x} = f(x) + G(x)u$. Suppose there exist a positive definite symmetric matrix P, a positive semidefinite function $W(x)$, and positive constants γ and σ such that

$$2x^TPf(x) + \gamma x^TPx + W(x) - 2\sigma x^TPG(x)G^T(x)Px \leq 0, \quad \forall\, x \in R^n$$

Show that with $u = -\sigma G^T(x)Px$ the closed-loop system has a globally exponentially stable equilibrium point at the origin.

4.44 Consider the system

$$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin t, \qquad \dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos t$$

Show that the origin is exponentially stable and estimate the region of attraction.

[37] See [158] for a comprehensive treatment of Floquet theory.

4.45 Consider the system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3, \qquad \dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0$, for all $t \geq 0$.

(a) Is the equilibrium point $x = 0$ uniformly asymptotically stable?

(b) Is it exponentially stable?

(c) Is it globally uniformly asymptotically stable?

(d) Is it globally exponentially stable?

4.46 Show that the origin of the system

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

is asymptotically stable. Is it exponentially stable?

4.47 Consider the system

$$\dot{x}_1 = -\phi(t)x_1 + a\phi(t)x_2, \qquad \dot{x}_2 = b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3$$

where a, b, and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall\, t \geq 0$$

Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

4.48 Consider two systems represented by $\dot{x} = f(x)$ and $\dot{x} = h(x)f(x)$ where $f : R^n \to R^n$ and $h : R^n \to R$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of the first system is exponentially stable if and only if the origin of the second system is exponentially stable.

4.49 Show that the system

$$\dot{x}_1 = -ax_1 + b, \qquad \dot{x}_2 = -cx_2 + x_1(\alpha - \beta x_1 x_2)$$

where all coefficients are positive, has a globally exponentially stable equilibrium point.
Hint: Shift the equilibrium point to the origin and use V of the form $V = k_1 Y_1^2 + k_2 y_2^2 + k_3 y_1^4$, where (y_1, y_2) are the new coordinates.

4.50 Consider the system

$$\dot{x} = f(t,x); \quad f(t,0) = 0$$

where $[\partial f/\partial x]$ is bounded and Lipschitz in x in a neighborhood of the origin, uniformly in t for all $t \geq t_0 \geq 0$. Suppose that the origin of the linearization at $x = 0$ is exponentially stable, and the solutions of the system satisfy

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, \|x(t_0)\| < c \tag{4.59}$$

for some class $\mathcal{KL}$ function β and some positive constant c.

(a) Show that there is a class $\mathcal{K}$ function α and a positive constant γ such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \exp[-\gamma(t - t_0)], \quad \forall\, t \geq t_0, \quad \forall\, \|x(t_0)\| < c$$

(b) Show that there is a positive constant M, possibly dependent on c, such that

$$\|x(t)\| \leq M\|x(t_0)\| \exp[-\gamma(t - t_0)], \quad \forall\, t \geq t_0, \quad \forall\, \|x(t_0)\| < c \tag{4.60}$$

(c) If inequality (4.59) holds globally, can you state inequality (4.60) globally?

4.51 Suppose the assumptions of Theorem 4.18 are satisfied with $\alpha_1(r) = k_1 r^a$, $\alpha_2(r) = k_2 r^a$, and $W(x) \geq k_3\|x\|^a$, for some positive constants k_1, k_2, k_3, and a. Show that (4.42) and (4.43) are satisfied with $\beta(r,s) = kr\exp(-\gamma s)$ and $\alpha_1^{-1}(\alpha_2(\mu)) = k\mu$, where $k = (k_2/k_1)^{1/a}$ and $\gamma = k_3/(k_2 a)$.

4.52 Consider Theorem 4.18 when $V(t,x) = V(x)$ and suppose inequality (4.40) is replaced by

$$\frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall\, W_4(x) \geq \mu > 0$$

for some continuous positive definite functions $W_3(x)$ and $W_4(x)$. Show that (4.42) and (4.43) hold for every initial state $x(t_0) \in \{V(x) \leq c\} \subset D$, provided $\{V(x) \leq c\}$ is compact and $\max_{W_4(x)\leq\mu} V(x) < c$.

4.53 (**[72]**) Consider the system $\dot{x} = f(t,x)$ and suppose there is a function $V(t,x)$ that satisfies

$$W_1(x) \leq V(t,x) \leq W_2(x), \quad \forall\, \|x\| \geq r > 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) < 0, \quad \forall\, \|x\| \geq r_1 \geq r$$

where $W_1(x)$ and $W_2(x)$ are continuous, positive definite functions. Show that the solutions of the system are uniformly bounded.
Hint: Notice that $V(t,x)$ is not necessarily positive definite.

4.54 For each of the following scalar systems, investigate input-to-state stability:

$$(1)\ \dot{x} = -(1+u)x^3 \qquad (2)\ \dot{x} = -(1+u)x^3 - x^5$$

$$(3)\ \dot{x} = -x + x^2 u \qquad (4)\ \dot{x} = x - x^3 + u$$

4.55 For each of the following systems, investigate input-to-state stability:

$$\begin{array}{llll}
(1) & \dot{x}_1 = -x_1 + x_1^2 x_2, & & \dot{x}_2 = -x_1^3 - x_2 + u \\
(2) & \dot{x}_1 = -x_1 + x_2, & & \dot{x}_2 = -x_1^3 - x_2 + u \\
(3) & \dot{x}_1 = x_2, & & \dot{x}_2 = -x_1^3 - x_2 + u \\
(4) & \dot{x}_1 = (x_1 - x_2 + u)(x_1^2 - 1), & & \dot{x}_2 = (x_1 + x_2 + u)(x_1^2 - 1) \\
(5) & \dot{x}_1 = -x_1 + x_1^2 x_2, & & \dot{x}_2 = -x_2 + x_1 + u \\
(6) & \dot{x}_1 = -x_1 - x_2 + u_1, & & \dot{x}_2 = x_1 - x_2^3 + u_2 \\
(7) & \dot{x}_1 = -x_1 + x_2, & & \dot{x}_2 = -x_1 - \sigma(x_1) - x_2 + u
\end{array}$$

where σ is a locally Lipschitz function, $\sigma(0) = 0$, and $y\sigma(y) \geq 0$ for all $y \neq 0$.

4.56 Using Lemma 4.7, show that the origin of the system

$$\dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = -x_2^3$$

is globally asymptotically stable.

4.57 Prove another version of Theorem 4.19, where all the assumptions are the same except that inequality (4.49) is replaced by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_3(\|x\|) + \psi(u)$$

where α_3 is a class $\mathcal{K}_\infty$ function and $\psi(u)$ is a continuous function of u with $\psi(0) = 0$.

4.58 Use inequality (4.47) to show that if $u(t)$ converges to zero as $t \to \infty$, so does $x(t)$.

4.59 Consider the scalar system $\dot{x} = -x^3 + e^{-t}$. Show that $x(t) \to 0$ as $t \to \infty$.

4.60 Suppose the assumptions of Theorem 4.19 are satisfied for $\|x\| < r$ and $\|u\| < r_u$ with class $\mathcal{K}$ functions α_1 and α_2 that are not necessarily class $\mathcal{K}_\infty$. Show that there exist positive constants k_1 and k_2 such that inequality (4.47) is satisfied for $\|x(t_0)\| < k_1$ and $\sup_{t \geq t_0} \|u(t)\| < k_2$. In this case, the system is said to be *locally input-to-state stable.*

4.61 Consider the system

$$\dot{x}_1 = x_1 \left\{ \left[\sin\left(\frac{\pi x_2}{2} \right) \right]^2 - 1 \right\}, \qquad \dot{x}_2 = -x_2 + u$$

(a) With $u = 0$, show that the origin is globally asymptotically stable.

(b) Show that for any bounded input $u(t)$, the state $x(t)$ is bounded.

(c) With $u(t) \equiv 1$, $x_1(0) = a$, and $x_2(0) = 1$, show that the solution is $x_1(t) \equiv a$, $x_2(t) \equiv 1$.

(d) Is the system input-to-state stable?

In the next seven exercises, we deal with the discrete-time dynamical system[38]

$$x(k+1) = f(x(k)), \quad f(0) = 0 \tag{4.61}$$

The rate of change of a scalar function $V(x)$ along the motion of (4.61) is defined by

$$\Delta V(x) = V(f(x)) - V(x)$$

4.62 Restate Definition 4.1 for the origin of the discrete-time system (4.61).

4.63 Show that the origin of (4.61) is stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite. Show that it is asymptotically stable if, in addition, $\Delta V(x)$ is negative definite. Finally, show that the origin is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.

4.64 Show that the origin of (4.61) is exponentially stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \qquad \Delta V(x) \le -c_3\|x\|^2$$

for some positive constants c_1, c_2, and c_3.
Hint: For discrete-time systems, exponential stability is defined by the inequality $\|x(k)\| \le \alpha\|x(0)\|\gamma^k$ for all $k \ge 0$, where $\alpha \ge 1$ and $0 < \gamma < 1$.

4.65 Show that the origin of (4.61) is asymptotically stable if, in a neighborhood of the origin, there is a continuous positive definite function $V(x)$ so that $\Delta V(x)$ is negative semidefinite and $\Delta V(x)$ does not vanish identically for any $x \neq 0$.

4.66 Consider the linear system $x(k+1) = Ax(k)$. Show that the following statements are equivalent:

[38] See [95] for a detailed treatment of Lyapunov stability for discrete-time dynamical systems.

(1) $x = 0$ is asymptotically stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) Given any $Q = Q^T > 0$, there exists $P = P^T > 0$, which is the unique solution of the linear equation $A^T PA - P = -Q$.

4.67 Let A be the linearization of (4.61) at the origin; that is, $A = [\partial f/\partial x](0)$. Show that the origin is asymptotically stable if all the eigenvalues of A have magnitudes less than one.

4.68 Let $x = 0$ be an equilibrium point for the nonlinear discrete-time system $x(k+1) = f(x(k))$, where $f : D \to R^n$ is continuously differentiable and $D = \{x \in R^n \mid \|x\| < r\}$. Let C, $\gamma < 1$, and r_0 be positive constants with $r_0 < r/C$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the solutions of the system satisfy

$$\|x(k)\| \le C\|x(0)\|\gamma^k, \quad \forall\ x(0) \in D_0, \ \forall\ k \ge 0$$

Show that there is a function $V : D_0 \to R$ that satisfies

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\Delta V(x) = V(f(x)) - V(x) \le -c_3\|x\|^2$$

$$|V(x) - V(y)| \le c_4\|x - y\|\,(\|x\| + \|y\|)$$

for all $x, y \in D_0$ for some positive constants c_1, c_2, c_3, and c_4.