V is radially unbounded. Thus

$y(t) \to 0$ as $t \to \infty$. Can

we conclude that $k(t) \to k_\infty$ some

specific value? Yes but the argument

is subtle and is dependent on real

analyticity of V!

Remark: In this problem, we have shown

that a suitable Lyapunov function exists

but we don't know it explicitly ($\delta > \frac{a}{b}$

is unknown since system is unknown)

$\boxed{\text{EXAMPLE 2}}$

We now show the connection between the

example just analyzed and the problem of

model reference adaptive control.

Consider a plant given by

$$\dot{y}_p = a_p y_p + k_p u$$

We have a reference model

$$\dot{y}_m = a_m y_m + k_m r$$

The adaptive controller is intended to make the plant behave like the reference model.

The controller $u(t) = \theta_1^* r(t) + \theta_2^* y_p(t)$

where $\theta_1^* = \frac{k_m}{k_p}$, and $\theta_2^* = \frac{a_m - a_p}{k_p}$

does the job, but we need to know k_p and a_p exactly. What if these are unknown?

Consider

$$u(t) = \theta_1(t)\, r(t) + \theta_2(t)\, y_p(t)$$

(a controller of the same form as before, but θ_i vary with time).

Then consider the "gradient rule", to be justified later,

$$\dot{\theta}_1 = -\gamma (y_p - y_m) r \qquad \gamma > 0$$

$$\dot{\theta}_2 = -\gamma (y_p - y_m) y_p$$

Assume that the <u>sign of k_p is known</u>, $k_p > 0$.

The state space model of the closed loop system takes the form below. First define

$e_0 \triangleq y_p - y_m$ (output error)

$\phi_1 \triangleq \theta_1 - \theta_1^*$ (parameter error)

$\phi_2 \triangleq \theta_2 - \theta_2^*$ (" ")

Then $\dot{y}_m = a_m y_m + k_m r$

$$= (a_p + \theta_2^* k_p) y_m + \theta_1^* k_p r$$

(by def^n of θ_1^*, θ_2^*)

$$= \boxed{a_p y_m + k_p(\theta_1^* r + \theta_2^* y_m)}$$

$$\dot{y}_p = \boxed{a_p y_p + k_p(\theta_1 r + \theta_2 y_p)}$$

$$\Rightarrow \quad \dot{e}_0 = \dot{y}_p - \dot{y}_m$$

$$= a_p e_0 + k_p(\theta_1 - \theta_1^*) r + k_p(\theta_2 y_p - \theta_2^* y_m)$$

(by differencing the boxed equations)

$$= a_p e_0 + k_p(\theta_1 - \theta_1^*) r + k_p(\theta_2 y_p - \theta_2^* y_m + \theta_2^* y_p - \theta_2^* y_p)$$

$$= \left(a_p e_0 + k_p \theta_2^* e_0\right) + k_p(\theta_1 - \theta_1^*) r + k_p(\theta_2 - \theta_2^*) y_p$$

$$\dot{e}_0 = a_m e_0 + k_p \cdot \phi_1 \cdot r + k_p \phi_2 \cdot (y_m + e_0)$$

(from definitions of θ_2^*, e_0, ϕ_1, ϕ_2)

So the closed loop system equations (both inner & outer loops are closed) are

$$\dot{e}_0 = a_m e_0 + k_p \phi_1 \cdot r + k_p \phi_2 \cdot (e_0 + y_m)$$

$$\dot{\phi}_1 = -\gamma e_0 r$$

$$\dot{\phi}_2 = -\gamma e_0 (e_0 + y_m)$$

where $\dot{\phi}_i = \dot{\theta}_i$ is used

The signals r, y_m, ϕ_p are time-varying.

There are $\underbrace{\text{three}}$ special cases.

(1) k_p is known $\Rightarrow$ $\theta_1 = \theta_1^*$ drop ϕ_1 equation and let $\phi \triangleq \phi_2$.

Then

$$\dot{e}_0 = a_m e_0 + k_p \cdot \phi \cdot (e_0 + y_m)$$

$$\dot{\phi} = -\gamma \cdot e_0 (e_0 + y_m)$$

(2) $r(t) \equiv 0 \implies y_m(t) \equiv 0$. (assume $y_m(0) = 0$)
k_p not known but $k_p > 0$.

$$\Rightarrow \quad \dot{e}_0 = (a_m + k_p \phi) e_0 ; \quad \dot{\phi} = -\gamma e_0^2$$

We are now in exactly the same situation as in EXAMPLE 1 and by letting

$$\Phi(e_0, \phi) = \gamma e_0^2 + k_p\left(\phi + \frac{a_m}{k_p}\right)^2$$

and observing that Φ is constant along the trajectories of the closed loop system we conclude (as in EXAMPLE 1) that $e_0 \to 0$ and $k_p \to k_\infty$ a constant

< Trajectories move on semi-ellipses backward >

The analysis of case (1) and the more general setting of EXAMPLE 2 needs to take account of time-dependent (nonautonomous) systems.

(3) In this third special case, we assume a_p is known $= a_m$. But k_p is unknown. We further assume $k_p > 0$ (sign of k_p is known). Then $\theta_2^* = 0$, no matter what k_p is, so it makes sense to ask $\theta_2 = 0 = \theta_2^*$. Thus ϕ_2 drops out

The closed loop equations are

$$\dot{e}_0 = a_m e_0 + k_p \phi \cdot r$$

$$\dot{\phi} = -\gamma e_0 \cdot r$$

$$\phi \triangleq \phi_1 = \theta_1 - \theta_1^*$$

We analyze this system via the (Lyapunov) function $V(e_0, \phi) = \gamma e_0^2 + k_p \phi^2$

Along the trajectories of the closed loop system

$$\frac{dV}{dt} = 2\gamma e_0 \dot{e}_0 + 2 k_p \phi \dot{\phi}$$

$$= 2\gamma e_0 (a_m e_0 + k_p \phi r)$$

$$+ 2k_p \phi (-\gamma e_0 r)$$

$$= 2\gamma a_m e_0^2$$

Suppose $a_p = a_m < 0$ (open loop stable)

Then $2\gamma a_m e_0^2 \leq 0$.

Thus we have V positive definite and $\dot{V} \leq 0$ along traj. of closed loop system.

Thus e_0 and ϕ are bounded.

<u>Do they go to 0?</u>

<u>Note</u>: closed loop system is non-autonomous ($r(t)$ is dependent on t) so we need a theorem that applies to nonautonomous o.d.e's.

If $r(t) =$ constant $= 0$, then parameter error is unchanged (no adaptation) but by hypothesis that $a_m < 0$, $e_0 \to 0$.

If $r(t) =$ constant $\neq 0$, then from

$$\ddot{e}_0 = a_m \dot{e}_0 + k_p \dot{\phi} r$$

$$= a_m \dot{e}_0 + k_p(-\gamma e_0 r) r$$

it follows that

$$\ddot{e}_0 - a_m \dot{e}_0 + \gamma k_p r^2 e_0 = 0$$

Since $a_m < 0$, $\gamma, k_p > 0$ it follows that $e_0(t) \to 0$ as $t \to \infty$ (oscillatory decay)

$$\ddot{\phi} = -\gamma \dot{e}_0 r$$

$$= -\gamma r (a_m e_0 + k_p \phi r)$$

$$= -\gamma r \left(a_m \frac{(-\dot{\phi}}{\gamma r})\right) - k_p \phi \gamma r^2$$

$$= + a_m \dot{\phi} - k_p \phi \gamma r^2$$

$$\Rightarrow \quad \ddot{\phi} - a_m \dot{\phi} + (k_p \gamma r^2) \phi = 0$$

again: oscillatory decay to 0 as $t \to \infty$.

One can see the convergence result more quickly by appealing to LaSalle's invariance principle:

$$\dot{e}_0 = a_m e_0 + k_p \phi \cdot r$$

$$\dot{\phi} = -\gamma e_0 \cdot r$$

$r \neq 0$
$\gamma > 0$
$a_m < 0$
$k_p > 0$

<u>Only</u> equilibrium is $(0, 0)$.