ENEE 661 Nonlinear Control Systems
Lecture 6 (part ii)

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We now consider the proof of a technical lemma used in the main theorem for stability of timevarying systems (part (i) of this lecture)

Lemma 1 Let $\dot{y} = -\alpha(y)$ $y(t_0) = y_0$ and $\alpha(\cdot)$ a class $\mathcal{K}$ function. Assume further that $\alpha(\cdot)$ is locally Lipschitz. Suppose α is defined on $[0, a]$. Then, for all $0 \le y_0 \le a$, the equation has a unique solution $y(t)$ defined $\forall\, t \geq t_0$. Moreover $y(t) = \sigma(y_0, t-t_0)$ where σ is a class $\mathcal{KL}$ function on $[a, a) \times [0, \infty)$.

Proof $\alpha(\cdot)$ is locally Lipschitz $\Rightarrow \exists!$ solution $\forall\, y_0 \geq 0$. Since $\dot{y}(t) < 0$ whenever $y(t) > 0$, the solution $y(t) \le y_0$ $\forall\, t \geq t_0$. Therefore the solution is bounded and can be extended $\forall\, t \geq t_0$.

By integration

$$\eta_{y_0}(y) \triangleq -\int_{y_0}^{y} \frac{dx}{\alpha(x)} = t - t_0$$

gives the sojourn time map (i.e. how long it takes to get to y from y_0) defined on $(0, y_0)$

Let $\eta(y) \triangleq \eta_b(y)$ $0 < b < a$.

$\eta(\cdot)$ is strictly decreasing, differentiable on $(0, a)$. Moreover $\eta(y) \to \infty$ as $y \to 0$.

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To see this, note that $y(t) \to 0$ as $t \to \infty$, since $\dot{y}(t) < 0$, for $y(t) > 0$. This can only happen asymptotically as $t \to \infty$; i.e it cannot happen in ~~a~~ finite time without violating uniqueness.

Notice that since $b < a$

$$\eta(a) = -\int_b^a \frac{dx}{\alpha(x)} = -c$$

for some $c > 0$

We have $\eta : (0, a) \to (-c, \infty)$ and $\eta^{-1} : (-c, \infty) \to (0, a)$ is also well defined, since η is a strictly decreasing function of its argument.

~~Then~~ Then, for $y_0 > 0$,

$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0)$$

and

$$y(t) \equiv 0 \quad \text{if} \quad y_0 = 0.$$

Define
$$\sigma(r, s) = \begin{cases} \eta^{-1}(\eta(r) + s) & r > 0 \\ 0 & r = 0 \end{cases}$$

Then $y(t) = \sigma(y_0, t - t_0) \quad \forall \; t \geq t_0, \; y_0 \geq 0$.

σ is continuous since both η and η^{-1} are continuous & $\lim_{x \to \infty} \eta^{-1}(x) = 0$

since $(t - t_0) = \eta_{y_0}(y) = -\int_{y_0}^{y} \frac{dx}{\alpha(x)} = -\left(\int_{y_0}^{b} \frac{dx}{\alpha(x)} + \int_b^y \frac{dx}{\alpha(x)}\right) = -\eta(y_0) + \eta(y)$, ∴
we can omit y_0, ∴ and $\eta(y) = \eta(y_0) + (t - t_0)$

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Show these formulas: a little bit of calculus

For fixed s, since $\frac{\partial}{\partial r} \sigma(r,s)$

$$= \frac{\partial}{\partial r} \left\{ \eta^{-1} (\eta(r) + s) \right\}$$

$$\longrightarrow \qquad = \frac{\alpha(\sigma(r,s))}{\alpha(r)} > 0,$$

it is strictly increasing in r.

For fixed r, since $\frac{\partial \sigma}{\partial s}(r,s)$

$$\longrightarrow \qquad = -\alpha(\sigma(r,s)) < 0$$

it is strictly decreasing in s.

Furthermore, $\sigma(r,s) \to 0$ as $s \to \infty$ since $\eta^{-1} \to 0$ as its argument $\to \infty$. So we have shown σ is $\mathcal{KL}$

▨

Examples: (a) $\alpha(y) = -\gamma y$, $\gamma > 0$. $\sigma(r,s) = r e^{-\gamma s}$

(b) $\alpha(y) = -ky^2$, $k > 0$

$$\eta(y) = -\int_b^y \frac{dx}{kx^2}$$

$$= \frac{1}{k}\left(\frac{1}{y} - \frac{1}{b}\right)$$

and $$\sigma(r,s) = \frac{1}{\frac{1}{r} + ks}$$

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In the setting of <u>linear</u> <u>time varying</u> systems, some of the ideas concerning uniform stability <u>coalesce</u> as in the following theorem

<u>Theorem</u> 2

Let $\dot{x}(t) = A(t)x(t)$ be a linear system with piecewise continuous coefficient matrix $A(t)$. Then, the origin is uniformly asymptotically stable iff

$$\|\Phi(t,t_0)\| \leq k e^{-\gamma(t-t_0)}$$

for some $k > 0$ and $\gamma > 0$.
[i.e uniform asymptotic stability is equivalent to exponential stability in linear systems].

<u>Proof</u>

<u>Sufficiency</u> : trivial.

<u>Necessity</u> : there exists $\beta(\cdot,\cdot)$ of class $\mathcal{KL}$ such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) \quad \forall\ t \geq t_0 \quad \forall\ x(t_0) \in \mathbb{R}^n$$

$$\|\Phi(t,t_0)\| \triangleq \max_{\|y\|=1} \|\Phi(t,t_0)y\|$$

$$\leq \max_{\|y\|=1} \beta(\|y\|, t-t_0)$$

($\because$ solution
starting at y
at $\underline{t_0}$ is $\overline{\Phi(t,t_0)y}$)

$$= \beta(1, t-t_0).$$

Since $\beta(1,s) \to 0$ as $s \to \infty$, there exists $T > 0$ such that $\beta(1, t) \le \frac{1}{e}$ $\forall\, t \geq T$. For any $t \geq t_0$, let N be the smallest positiv integer s.t. $t \le t_0 + NT$. Divide the interval, $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each. Using the transition property of $\Phi(t, t_0)$ we can write,

$$\Phi(t,t_0) = \Phi(t, t_0+(N-1)T) \cdot \Phi(t_0+(N-1)T,\ t_0+(N-2)$$

$$\cdots\ \Phi(t_0+T, t_0)$$

Then

$$\|\Phi(t,t_0)\| \leqq \|\Phi(t, t_0+(N-1)T)\| \cdot \prod_{k=1}^{N-1} \|\Phi(t_0+kT, t_0+(k-1)T$$

$$\le \beta(1,0) \cdot \left(\frac{1}{e}\right)^{N-1}$$

$$\leqq e\,\beta(1,0)\, e^{-\frac{t-t_0}{T}}$$
$$= k\, e^{-\gamma(t-t_0)}$$

where $k \triangleq e\,\beta(1,0)$; $\gamma = \frac{1}{T}$. ▨

Remark. For time varying linear system there are no simple tests based on eigenvalues, to ascertain stability. One needs to use this theorem. However in case A is periodic in t, the Floquet-Lyapunov theorem does give a test for uniform asymptotic stability, namely: if $T =$ period of A &

all eigenvalues of $\Phi(T,0)$ are inside the open unit disk $\{z : |z| < 1\}$ in the complex plane, then we have uniform asymptotic stability (See Theorems 4 and Corollary 5 below)

In the remainder of this section of the notes we discuss two topics of importance — (i) the existence of Lyapunov functions for systems that demonstrate asymptotic stability properties — the so-called Lyapunov theorems; (ii) the indirect method of Lyapunov to assess stability of solutions of nonlinear (time varying) systems by assessing the stability of zero solutions of corresponding (time-varying) linearizations.

Theorem 3 Let $x = 0$ be an uniformly asymptotically stable equilibrium of $\dot{x}(t) = A(t)x(t)$. Let $A(t)$ be continuous, $\|A(t)\|_2 \leq L$ $\forall$ $t \geq 0$. Let $Q(t)$ be continuous, symmetric positive definite such that, for suitable constants c_3 and c_4

$$0 < c_3 \mathbb{1} \leq Q(t) < c_4 \mathbb{1} \quad \forall t \geq 0.$$

Then there exists a unique symmetric positive definite $P(t)$ satisfying

$$-\dot{P} = A^T P + PA + Q$$

and $P > 0$ is bounded above and below,

$$0 < c_1 \mathbb{1} < P(t) < c_2 \mathbb{1} \qquad \forall\, t \geq 0$$

for suitable constants c_1 & c_2. Hence $V(t,x) = x^T P(t) x$ is a time varying Lyapunov function for the given linear system, in the sense of Theorem 1.

Proof: First recall that the notation

$$a \mathbb{1} \leq M \leq b \mathbb{1}$$

means

$$a\, y^T y \leq y^T M y \leq b\, y^T y \qquad \forall\, y \in \mathbb{R}^n$$

$$\leftrightarrow \quad a \leq \frac{y^T M y}{y^T y} \leq b$$

$$\leftrightarrow \quad a \leq \lambda_{min}(M) \leq \lambda_{max}(M) \leq b .$$

Now define (Φ = transition matrix of A)

$$P(t) = \int_t^{\infty} \Phi^T(\tau, t)\, Q(\tau)\, \Phi(\tau, t)\, d\tau$$

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It is easy to check that $P(t)$ is the only solution of

$$-\dot{P} = A^T P + PA + Q$$

(A, P, Q all depend on time t).

Let $\phi(\tau, t, x)$ denote the solution to the given linear system starting at x at time t. Then by linearity,

$$\phi(\tau, t, x) = \Phi(\tau, t) x .$$

Then,

$$
\begin{aligned}
V(t, x) &= x^T P(t) x \\
&= x^T \left(\int_t^\infty \Phi^T(\tau, t) \, Q(\tau) \, \Phi(\tau, t) \, d\tau \right) x \\
&= \int_t^\infty \phi^T(\tau, t, x) \, Q(\tau) \, \phi(\tau, t, x) \, d\tau \\
&\le \int_t^\infty c_4 \|\phi(\tau, t, x)\|_2^2 \, d\tau && (\text{by hyp. on } Q) \\
&\le \int_t^\infty c_4 \, \|\Phi(\tau, t)\|_2^2 \, \|x\|_2^2 \, d\tau \\
&\le \int_t^\infty c_4 \cdot k^2 e^{-2\gamma(\tau - t)} \, \|x\|_2^2 \, d\tau && (\text{by thm 2 and hyp. on system}) \\
&= \frac{k^2 c_4}{2\gamma} \|x\|_2^2 \triangleq c_2 \|x\|_2^2 && \forall t \geq 0
\end{aligned}
$$

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On the other hand, since $\|A(t)\|_2 \leq L \quad \forall t \geq 0$, by hypothesis, one can show that

an early homework problem $\longleftarrow$

$$\frac{d}{dt}\|\phi(\tau,t,x)\|_2^2 \geq -2L\,\|\phi(\tau,t,x)\|_2^2$$

$$\Rightarrow \quad \|\phi(\tau,t,x)\|_2^2 \geq \|\cancel{\phi}x\|_2^2\, e^{-2L(\tau-t)}.$$

Hence,

$$V(t,x) \geq \int_t^\infty c_3\,\|\phi(\tau,t,x)\|_2^2\, d\tau$$

$$\geq \int_t^\infty c_3\, e^{-2L(\tau-t)}\,\|x\|_2^2\, d\tau$$

$$= \frac{c_3}{2L}\|x\|_2^2 = c_1\,\|x\|_2^2 .$$

Thus $c_1\|x\|_2^2 \leq V(t,x) = x^T P(t) x \leq c_2\,\|x\|_2^2$ with

$$c_1 = \frac{c_3}{2L} \quad \text{and} \quad c_2 = \frac{k^2 c_4}{2\gamma}.$$

Furthermore,

$$\dot{V} = \frac{\partial}{\partial t}V(t,x) + \frac{\partial V}{\partial x}\cdot A(t)x$$

$$= x^T(\dot{P} + A^T P + PA)x$$

$$= -x^T Q(t)\, x$$

$$\leq -c_3\,\|x\|_2^2$$

Thus $V(t,x) = x^T P(t) x$ is a time-dependent Lyapunov function in the sense of theorem 1 satisfying

$$\alpha_1(\|x\|_2) \leq V(t,x) \leq \alpha_2(\|x\|)$$

$$\dot{V}(t,x) \leq -\alpha_3(\|x\|)$$

where the class K functions α_i are given by

$$\alpha_i(y) = c_i(y)^2$$

with $c_1 = \frac{c_3}{2L}$; $c_2 = \frac{c_4 k^2}{2\gamma}$; c_3 given

▨

Remark The formula

$$V(t,x) = \int_t^\infty \phi(\tau,t,x)^T Q(\tau)\, \phi(\tau,t,x)\, d\tau$$

suggests a possible path to converse Lyapunov theorems for nonlinear systems — let $\phi(\tau,t,x)$ be the solution starting at x at t for the nonlinear system.

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Theorem 4 (Periodic Linear Systems)

Consider $\dot{x}(t) = A(t)\,x(t)$, $A(\cdot)$ piecewise continuous, $x(t_0) = x_0$, $A(t+T) = A(t) \quad \forall\, t$

Let Φ denote the transition matrix,

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)$$

$$\Phi(t_0,t_0) = 1\!\!1$$

Then

(a) $\Phi(t+T,\, t_0+T) = \Phi(t,t_0)$

(b) There is a constant matrix R and a T-periodic nonsingular matrix function $P(t)$ such that

$$\Phi(t,t_0) = P^{-1}(t)\, e^{R(t-t_0)}\, P(t_0)$$

(c) 0 is uniformly (asymptotically) stable for the given system iff it is uniformly (asymptotically) stable for the system

$$\dot{z} = Rz.$$

Proof : (a) $\Phi(t+T,\, t_0+T) = 1\!\!1 + \int_{t_0+T}^{t+T} A(\sigma_1)\,d\sigma_1$

$$+ \int_{t_0+T}^{t+T}\int_{t_0}^{\sigma_1} A(\sigma_1)A(\sigma_2)\,d\sigma_2\,d\sigma_1$$

$$+ \cdots$$

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$$= \mathbb{1} + \int_{t_0+T}^{t+T} A(\sigma_1+T)\, d\sigma_1 + \int_{t_0+T}^{t+T} \int_{t_0+T}^{\sigma_1} A(\sigma_1+T) A(\sigma_2+T)\, d\sigma_2\, d\sigma_1$$

$$+ \cdots$$

(by T- periodicity of A)

$$= \mathbb{1} + \int_{t_0}^{t} A(\sigma_1)\, d\sigma_1 + \int_{t_0}^{t} \int_{t_0}^{\sigma_1} A(\sigma_1) A(\sigma_2)\, d\sigma_2\, d\sigma_1 + \cdots$$

(by change of variables $\sigma_1 \leftarrow \sigma_1 + T$)

$$= \Phi(t, t_0)$$

(b) $\Phi(T,0)$ is nonsingular and hence has a (possibly complex matrix) logarithm RT , i.e.

$$\Phi(T,0) = e^{RT} .$$

Let $P^{-1}(t) \triangleq \Phi(t,0) e^{-Rt}$ $\quad (P(0) = \mathbb{1})$

$$P^{-1}(t+T) = \Phi(t+T, 0) e^{-R(t+T)}$$

$$= \Phi(t+T, T)\, \Phi(T,0)\, e^{-RT} e^{-Rt}$$

$$= \Phi(t,0) e^{-Rt} \qquad \text{(by (a))}$$

$$= P^{-1}(t) .$$

Thus $\Phi(t, t_0) = \Phi(t,0)\, \Phi(0, t_0)$

$$= \Phi(t,0) \left(\Phi(t_0, 0)\right)^{-1}$$

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$$= P^{-1}(t)e^{+Rt}\ (P^{-1}(t_0)\,e^{+Rt_0})^{-1}$$

$$= P^{-1}(t)\,e^{Rt}\,e^{-Rt_0}\,P(t_0)$$

$$= P^{-1}(t)\,e^{R(t-t_0)}\,P(t_0)$$

(c) Let $z(t) = P(t)\,x(t)$

$$\dot{z} = \dot{P}\,x + P\dot{x}$$

$$= \dot{((P^{-1})^{-1})}\ x + P\dot{x}$$

$$= -P\,\dot{P^{-1}}\,P\,x + P\dot{x}$$

$$= -P\,\dot{(\Phi(t,0)e^{-Rt})}\,Px + P\dot{x}$$

$$= -\Big(P(t)A(t)\Phi(t,0)\,e^{-Rt}P(t) + P(t)\Phi(t,0)\,e^{-Rt}(-R)P\Big)x + PAx$$

$$= -PAx + (+R)Px + PAx$$
$$= +Rz$$

P has piecewise continuous derivative on $(-\infty, \infty)$
$\dot{P} = RP - PA$; P and $\dot{P}$ are bounded on $(-\infty,\infty)$ because there are piecewise continuous and T-periodic; $\exists\ m_1, m_2 > 0$ s.t. $0 < m_1 \le |\det P(t)| \le m_2$ by these properties.

Hence $\|z(t)\| < c_1 \|x(t)\|$

and $\|x(t)\| < c_2 \|z(t)\|$

where
$$c_1 = \max_{[t,\, t+T]} \|P(t)\|$$

$$c_2 = \max_{[t,\, t+T]} \|P^{-1}(t)\| .$$

From these two inequalities it follows that all the stability properties of z carry over to those of x & vice versa. ▨

Corollary 5 Let $A(t+T) = A(t)$ be piecewise continuous and T-periodic. Let $x=0$ be uniformly, asymptotically, stable equilibrium of $\dot{x}(t) = A(t)x(t)$. Then there is a Lyapunov function $V = V(t,x) = V(t+T, x) = x^T P(t) x$ satisfying
$$-\dot{P} = A^T P + PA + Q$$
for each Q T-periodic and
$$0 < c_3 \mathbb{1} \le Q(t) \le c_4 \mathbb{1} .$$

Proof Essentially same construction as in Theorem 3 ☒

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Example (Floquet-Lyapunov)

$$A(t) = \begin{pmatrix} -1+\cos(t) & 0 \\ 0 & -2+\cos(t) \end{pmatrix}$$

A (t) is 2π-periodic.

$$\Phi(2\pi, 0) = \begin{pmatrix} \exp\left(\int_0^{2\pi}(-1+\cos t)\,dt\right) & 0 \\ 0 & \exp\left(\int_0^{2\pi}(-2+\cos(t))\,dt\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2\pi} & 0 \\ 0 & e^{-4\pi} \end{pmatrix}.$$

has both eigenvalues in the open unit disk.

$\Rightarrow$ 0 is uniformly asymptotically stable.