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ENEE 765 Lecture 5 (deterministic identification)

Example 1 $\dot{y}_p = -a_p y_p + k_p r$

plant transfer function $P(s) = \dfrac{k_p}{s + a_p}$

and, k_p and a_p are unknown.

Problem : Estimate k_p, a_p from data.

Naive approach

Take 2 measurements y_p, $\dot{y}_p$ and r at two different times t_1, t_2.

$$\begin{bmatrix} \dot{y}_p(t_1) \\ \dot{y}_p(t_2) \end{bmatrix} = \begin{bmatrix} y_p(t_1) & r(t_1) \\ y_p(t_2) & r(t_2) \end{bmatrix} \begin{bmatrix} -a_p \\ k_p \end{bmatrix}$$

This system of two linear equations can be solved to write

$$\begin{bmatrix} -a_p \\ k_p \end{bmatrix} = \begin{bmatrix} y_p(t_1) & r(t_1) \\ y_p(t_2) & r(t_2) \end{bmatrix}^{-1} \begin{bmatrix} \dot{y}_p(t_1) \\ \dot{y}_p(t_2) \end{bmatrix}$$

if the matrix inverse exists.

This gives too much importance to instants t_1 and t_2, ignores available time history of data, and requires time derivatives of plant output. One can avoid derivative measurements by using filtered derivatives. Denote $\hat{y}_p$ and $\hat{r}_p$ as Laplace transforms of y_p and r_p. Then,

$$(s + a_p)\, \hat{y}_p(s) = k_p\, \hat{r}(s) \qquad \text{ignoring } y_p(0).$$

Then $$\frac{s + a_p}{s + \lambda}\, \hat{y}_p(s) = \frac{k_p}{s+\lambda}\, \hat{r}(s) \qquad \text{for any } \lambda.$$

$$\Leftrightarrow \quad \frac{s+\lambda - (\lambda - a_p)}{s+\lambda}\, \hat{y}_p(s) = \frac{k_p}{s+\lambda}\, \hat{r}(s)$$

$$\Leftrightarrow \quad \hat{y}_p(s) = \frac{\lambda - a_p}{s+\lambda}\, \hat{y}_p(s) + \frac{k_p}{s+\lambda}\, \hat{r}(s)$$

$$= \underbrace{(\lambda - a_p)}_{\theta_2^*}\, \hat{w}^{(2)}(s) + \underset{\theta_1^*}{k_p}\, \hat{w}^{(1)}(s).$$

Here $\hat{w}^{(1)}(s)$ and $\hat{w}^{(2)}(s)$ are outputs of filters with transfer functions $\frac{1}{s+\lambda}$ and respective inputs $\hat{r}(s)$ and $\hat{y}_p(s)$.

Choosing $\lambda > 0$ (sufficiently large) makes transients in $w^{(1)}$ and $w^{(2)}$ die out (rapidly) as $t \to \infty$. One can write, in time domain,

$$y_p(t) = \underset{\theta_1^*}{k_p}\, w^{(1)}(t) + \underbrace{(\lambda - a_p)}_{\theta_2^*}\, w^{(2)}(t).$$

Sample filter outputs at times t_1 and t_2.

Now we can write

$$\begin{bmatrix} k_p \\ \lambda - a_p \end{bmatrix} = \begin{bmatrix} w^{(1)}(t_1) & w^{(2)}(t_1) \\ w^{(1)}(t_2) & w^{(2)}(t_2) \end{bmatrix}^{-1} \begin{bmatrix} y_p(t_1) \\ y_p(t_2) \end{bmatrix}$$

assuming the matrix inverse above exists. No derivatives are needed! Since λ is known one can determine k_p and a_p. Again it is desirable to use history of data instead of dealing with isolated time instants t_1 and t_2. We now lay out the ideas entirely in the time domain by setting

$$\theta^* = \begin{pmatrix} k_p \\ \lambda - a_p \end{pmatrix}$$

as ideal/nominal parameter vector. Let $\theta(t)$ be a <u>running guess</u> of θ^*. Then

$$\begin{aligned} e_1(t) &= \theta^T(t)\, w(t) - y_p(t) \\ &= \left(\theta^T(t) - \theta^{*T}\right) w(t) \\ &= \left(\theta(t) - \theta^*\right)^T w(t) = w^T(t)\left(\theta(t) - \theta^*\right) \end{aligned}$$

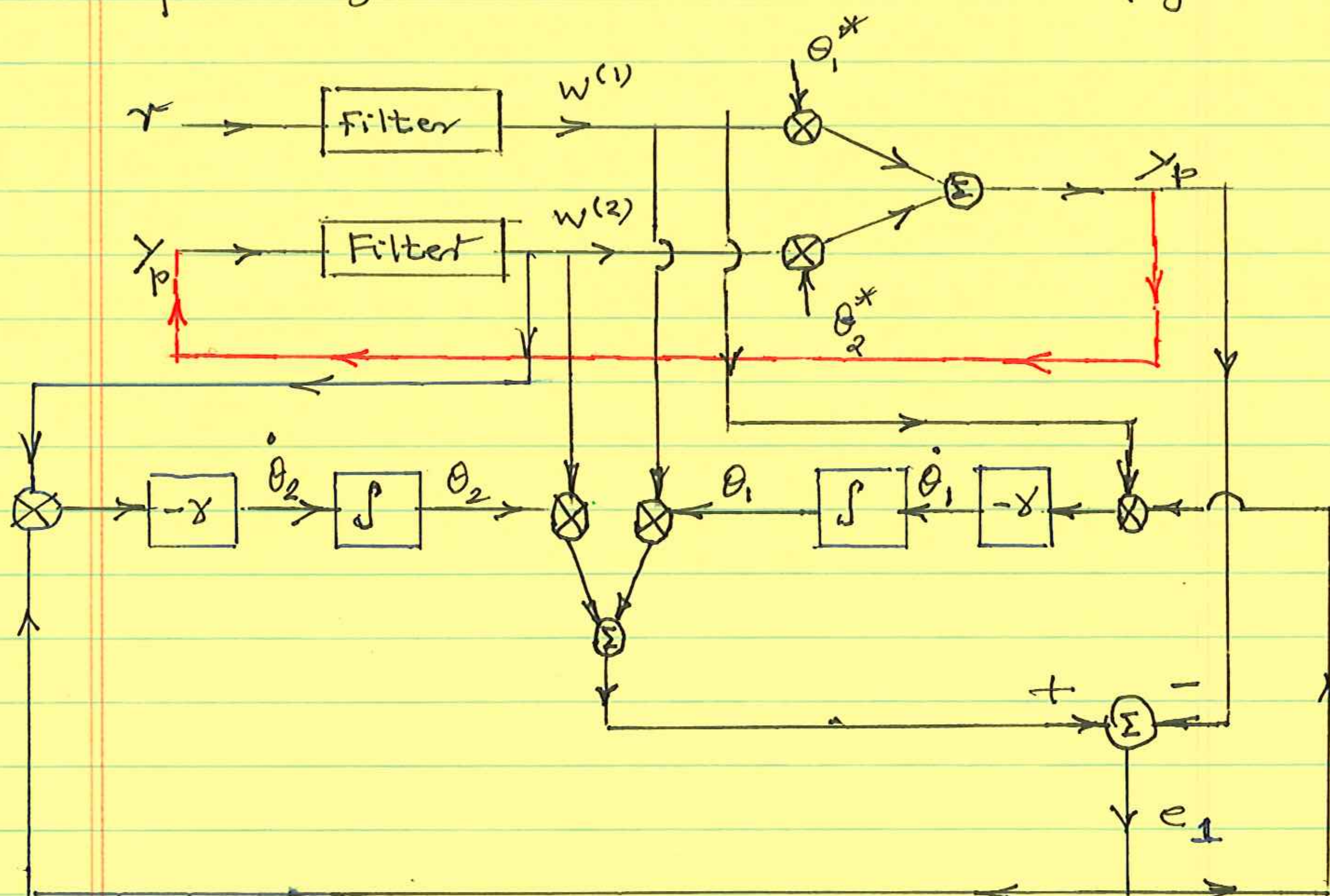
is the <u>identification error</u>, where

$$w(t) = \left(w^{(1)}(t) \quad w^{(2)}(t)\right)^T$$

We propose an on-line identification algorithm:

$$\dot{\theta}(t) = -\gamma\, w(t)\, e_1(t)$$

captured by the architecture in the figure below:



(6 multipliers and 3 summers)

Summarizing the above

Plant: $y_p = w^T \theta^*$; Filter: $\frac{1}{s+\lambda}$

Filter outputs: $w = \left(w^{(1)}\ \ w^{(2)}\right)^T$

Identifier: $\dot{\theta} = -\gamma\, w\, e_1$

Identification error $e_1 = w^T(\theta - \theta^*)$

The identifier is obeying a gradient rule since

$$\dot\theta = -\gamma e_1 w$$
$$= -\gamma e_1 \frac{\partial e_1}{\partial \theta} \qquad \gamma > 0$$
$$= -\gamma \frac{\partial}{\partial \theta}\left(e_1^2/2 \right).$$

We can rewrite

$$\dot\theta = -\gamma \left(\theta^T w - y_p\right) w$$
$$= -\gamma \left(\theta^T w - \theta^{*T} w\right) w$$
$$= -\gamma w \left(w^T (\theta - \theta^*)\right)$$

$$\Rightarrow \quad \dot\phi = \frac{d}{dt}(\theta - \theta^*)$$
$$= \dot\theta \qquad \text{since } \theta^* \text{ is constant}$$
$$= -\gamma w w^T \phi$$
$$= -\gamma \frac{\partial}{\partial \phi}\left(\frac{1}{2} \phi^T w w^T \phi\right)$$

a degenerate gradient descent equation since the quadratic form $\frac{1}{2}\phi^T w w^T \phi$ has rank 1.

Refinement

The gradient rule discussed above is greedy, it simply tries to lower an instantaneous output error, $e_1(t)$, instead of taking account of history, such as the integral squared error:

$$e_2(t) = \int_0^t \frac{1}{2}\left(\theta^T(t)\, w(\tau) - y_p(\tau)\right)^2 d\tau$$

note argument

$$\frac{\partial e_2}{\partial \theta(t)} = \left(\int_0^t w(\tau)\, w^T(\tau)\, d\tau\right)\theta(t) - \int_0^t y_p(\tau)\, w(\tau)\, d\tau$$

Minimization of $e_2(t)$ requires setting

$$\frac{\partial e_2}{\partial \theta(t)} = 0$$

Solving this, set

$$\theta_{LS}(t) = \left(\int_0^t w(\tau)\, w^T(\tau)\, d\tau\right)^{-1} \int_0^t y_p(\tau)\, w(\tau)\, d\tau$$

assuming the inverse exists. To compute recursively with time t, derive a differential equation for $\theta_{LS}(t)$. First, define

$$P(t) \triangleq \left(\int_0^t w(\tau)\, w^T(\tau)\, d\tau\right)^{-1}$$

Then $\frac{d}{dt} P^{-1}(t) = w(t)\, w^T(t)$

Also

$$0 = \frac{d}{dt}\left(P(t)\,P^{-1}(t)\right)$$

$$= \dot{P}P^{-1} + P\dot{P}^{-1}$$

$$\Rightarrow \quad \dot{P} = -P\dot{P}^{-1}P$$

$$= -P(t)\,w(t)\,w^T(t)\,P(t)$$

$$\dot{\theta}_{LS} = \frac{d}{dt}\left(P(t)\int_0^t w(\tau)\,y_p(\tau)\,d\tau\right)$$

$$= \dot{P}\cdot\int_0^t w(\tau)\,y_p(\tau)\,d\tau + P\,w(t)\,y_p(t)$$

$$= -P(t)\,w(t)\,w^T(t)\,P(t)\int_0^t w(\tau)\,y_p(\tau)\,d\tau$$

$$+ P(t)\,w(t)\,y_p(t)$$

$$= -P(t)\,w(t)\,w^T(t)\,\theta_{LS}(t) + P(t)\,w(t)\,y_p(t)$$

$$= -P(t)\,w(t)\left(w^T(t)\,\theta_{LS}(t) - y_p(t)\right)$$

$$= -P(t)\,w(t)\,e_1(t)$$

$$= -P(t)\,w(t)\,w^T(t)\,\phi_{LS}(t)$$

where $\phi_{LS}(t) = \theta_{LS}(t) - \theta^*$

$$\Rightarrow \dot{\phi}_{LS} = \dot{\theta}_{LS}(t) = -P(t)\,\frac{\partial}{\partial\phi}\left(\frac{1}{2}\,\phi_{LS}^T(t)\,w(t)\,w^T(t)\,\phi_{LS}(t)\right)$$

again a degenerate gradient descent with re-direction given by $P(t)$.

For the differential equations for $\dot{P}$ and $\dot{\theta}_{LS}$ to exactly capture P and θ_{LS}, they have to be initialized for some $t_0 > 0$ such that

$$P(t_0) = \left(\int_0^{t_0} w(\tau) w^T(\tau)\, d\tau \right)^{-1} \text{ exists.}$$

and

$$\theta_{LS}(t_0) = P(t_0) \int_0^{t_0} w(\sigma) y_p(\sigma)\, d\sigma$$

In practice one would initialize at $t=0$ with arbitrary (admissible) initial conditions, and verify that the effect of such a choice decays (rapidly) as $t \to \infty$.

Summarizing, the integral squared error criterion yields,

$$\dot{\theta}(t) = -P(t) w(t) \left(\theta^T(t) w(t) - y_p(t)\right)$$
$$\theta(0) = \theta_0$$
$$\dot{P}(t) = -P(t) w(t) w^T(t) P(t)$$
$$P(0) = P_0 = P_0^T > 0$$

with solutions $P(t) = \left(P_0^{-1} + \int_0^t w(\sigma) w^T(\sigma)\right)^{-1}$

and $\theta(t) = P(t) \left(P_0^{-1}\theta_0 + \int_0^t w(\sigma) y_p(\sigma)\, d\sigma\right)$

Parameter error

$$\phi(t) = \theta(t) - \theta^*$$

$$= \left(P_0^{-1} + \int_0^t w(\sigma)w^T(\sigma)\,d\sigma\right)^{-1} P_0^{-1}\phi(0)$$

$\phi(t) \to 0$ as $t \to \infty$ <u>if</u>

$$\int_0^t w(\sigma)w^T(\sigma)\,d\sigma \to \infty \quad \text{as } t \to \infty$$

(persistent excitation)

Comparison of Parameter Update Rules

INSTANTANEOUS/Greedy algorithm	INTEGRAL cost minimization
$\dot{\theta} = -\gamma e_1 w$	$\dot{\theta} = -Pwe_1$
$e_1 = \theta^T w - y_p$	$\dot{P} = -Pww^TP$
$\gamma > 0$	$P(0) = P^T(0) = P_0 > 0$
	$\theta(0) = \theta_0$

Generalizing Example 1.

plant: $P(s) = n_p(s) / d_p(s)$; $d_p(s)\,\hat{y}_p(s) = n_p(s)\,\hat{r}(s)$

$$n_p(s) = \alpha_n s^{n-1} + \alpha_{n-1} s^{n-2} + \dots + \alpha_1$$

$$d_p(s) = s^n + \beta_n s^{n-1} + \dots + \beta_1$$

and $(n_p(s), d_p(s)) \equiv 1$ (co-primeness).

Reference input $r(\cdot)$ bounded and piecewise continuous on $\mathbb{R}_+$.

goal: identify α_j, β_j $j = 1, 2, \dots, n$.

Assume $\lambda(s) = s^n + \lambda_n s^{n-1} + \dots + \lambda_1$
has all eigenvalues in open l.h.p (HURWITZ)

$$\frac{d_p(s)}{\lambda(s)}\,\hat{y}_p(s) = \frac{n_p(s)}{\lambda(s)}\,\hat{r}(s)$$

$$\frac{\lambda(s) - (\lambda(s) - d_p(s))}{\lambda(s)}\,\hat{y}_p(s) = \frac{n_p(s)}{\lambda(s)}\,\hat{r}(s)$$

$$\Rightarrow \hat{y}_p(s) = \frac{n_p(s)}{\lambda(s)}\,\hat{r}(s) + \frac{\lambda(s) - d_p(s)}{\lambda(s)}\,\hat{y}_p(s)$$

$$= \frac{a^*(s)}{\lambda(s)} + \frac{b^*(s)}{\lambda(s)}\,\hat{y}_p(s)$$

where $\deg(a_1^*(s)) \leq n-1$ <u>and</u>

$\deg(b^*(s)) \leq n-1$.

$$\text{Let} \quad \frac{b^*(s)}{\lambda(s)} = b^{*T}(s\mathbb{1} - \Lambda)^{-1} b_\lambda$$

$$\frac{a^*(s)}{\lambda(s)} = a^{*T}(s\mathbb{1} - \Lambda)^{-1} b_\lambda$$

where $a^* = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T$,

$$b^* = (\lambda_1 - \beta_1, \lambda_2 - \beta_2, \cdots, \lambda_n - b_n)^T$$

and
$$\Lambda = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & \\ & & & \\ & & 0 & 1 \\ -\lambda_1, & \cdots, & \lambda_{n-1} & \lambda_n \end{bmatrix} ; \quad b_\lambda = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

We have controllable but possibly unobservable triples $[\Lambda, b_\lambda, b^{*T}]$ and $[\Lambda, b_\lambda, a^{*T}]$ with associated dynamics

$$\dot{w}_p^{(1)} = \Lambda w_p^{(1)} + b_\lambda r$$

$$\dot{w}_p^{(2)} = \Lambda w_p^{(2)} + b_\lambda y_p$$

and output

$$y_p(t) = a^{*T} w_p^{(1)}(t) + b^{*T} w_p^{(2)}(t)$$
$$= \theta^{*T} w_p(t)$$

where $\theta^{*T} = (a^{*T}, b^{*T})$; $w_p = (w_p^{(1)T}, w_p^{(2)T})^T$

We have just written 2n dimensional state space realization of the given plant $P(s)$ of McMillan degree n. It is thus necessarily a nonminimal (unobservable) realization of the plant.

But all ~~poles of~~ unobservable modes of this realization are those associated to $\lambda(s)$, and hence stable.

Identifier Structure

Define

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda r \qquad (*)$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y$$

We claim:

(i) $(*)$ is an asymptotic observer for the 2n dimensional nonminimal realization above (without knowing θ^*).

(ii) for suitable hypothesis on $w(\cdot)$ and update rule for unknown parameter θ, parameter converges to zero.

proof of Claim (i)

$$\dot{\overbrace{(w^{(1)} - w_p^{(1)})}} = \Lambda (w^{(1)} - w_p^{(1)})$$

$$\dot{\overbrace{(w^{(2)} - w_p^{(2)})}} = \Lambda (w^{(2)} - w_p^{(2)})$$

Since $\lambda(s)$ is Hurwitz $(w^{(i)} - w_p^{(i)}) \to 0$ as $t \to \infty$, for $i = 1, 2$. $\square$

Regarding claim (ii)

Let $\theta^T(t) = (a^T(t), b^T(t))$

$$w^T(t) = (w^{(1)T}(t), w^{(2)T}(t))$$

Notice that we can write

$$y_p(t) = \theta^{*T} w_p(t)$$

$$= \theta^{*T} w(t) + (\theta^{*T} w_p(t) - \theta^{*T} w(t))$$

$$= \theta^{*T} w(t) - \varepsilon(t)$$

where $\varepsilon(t) \to 0$ exponentially as $t \to \infty$ (from claim (i))

Output of Identifier

$$y_i(t) = \theta^T(t) \underbrace{w(t)}_{\text{regressor vector}}$$

Identifier error

$$e_1(t) = y_i(t) - y_p(t)$$

$$= \theta^T(t) w(t) - (\theta^{*T} w(t) - \varepsilon(t))$$

$$= (\theta(t) - \theta^*)^T w(t) + \varepsilon(t)$$

$$= \underbrace{\phi^T(t)}_{\text{parameter error}} w(t) + \varepsilon(t)$$

In certain settings it is customary to assume $\varepsilon(t) \equiv 0$. This is not a bad approximation since $\varepsilon(t) \to 0$ exponentially as $t \to \infty$.

<u>Algorithm for identification</u> (parameter update rule)

The rules considered generalize naturally to the present settings as

(a) instantaneous/greedy <u>gradient algorithm</u>

(b) integral cost / <u>least squares algorithm</u>

(a)
$$\dot{\theta} = -\gamma \frac{\partial}{\partial \theta}\left(\frac{1}{2} e_1^2\right) = -\gamma e_1 \frac{\partial e_1}{\partial \theta} = -\gamma e_1 w$$

evolving in $\mathbb{R}^{2n}$, $\gamma > 0$

(b)
$$\dot{\theta} = -\gamma P w e_1$$
$$\dot{P} = Q - \gamma P w w^T P$$

$\gamma > 0$, $Q = Q^T > 0$, $P(0) = P^T(0) > 0$

evolving in $\mathbb{R}^{2n} \times \mathbb{R}^{2n(2n+1)/2}$

Both algorithms involve the "correlation" $w(t)\, e_1$ as a driving signal.

By correspondence with the Kalman filter, we refer to ~~[illegible]~~ P as a covariance matrix of parameter estimation. The least squares algorithm is more complicated to implement but one expects it to show faster convergence