2017.08.30 ENEE 765 Lecture 1

example $\dot{y} = ay + bu$ a, b constants

$u = -ky + r$ (inner loop)

reference input

when $r \equiv 0$, solution to closed-loop system:

$$y(t) = e^{t(a-bk)} y(0)$$

$\rightarrow 0$ as $t \rightarrow \infty$ if $a - bk < 0$

$\rightarrow \pm\infty$ as $t \rightarrow \infty$ if $a - bk > 0$

Condition for asymptotic stability: $a - bk < 0$, can be ensured by choice of k if a and b are known. If $a > 0$ we need inner loop, i.e. we need $k \neq 0$.

Suppose a and b are not known but it is known that $b > 0$. Then choose k large enough for asymptotic stability

Suppose it is known that $b < 0$. Choosing k sufficiently large in magnitude and negative ensures asymptotic stability.

Trial-and-error in choice of k is not the answer. Adaptation of k using knowledge of y is a systematic process:

$$\dot{k} = \beta y^2 \quad (\text{outer loop})$$

where β is a constant.

With $r \equiv 0$ the system with inner and outer loops:

$$(*) \qquad \begin{aligned} \dot{y} &= (a - bk)y \\ \dot{k} &= \beta y^2 \end{aligned}$$

is a nonlinear 2-dimensional system. Every point $(0, k)$, k arbitrary, is an equilibrium point of this system $(*)$.

The function

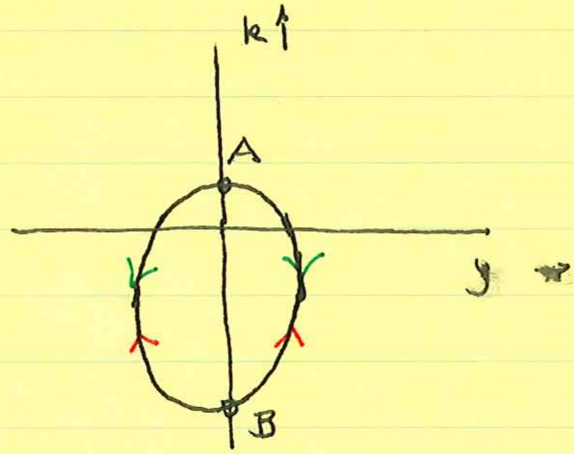
$$\phi(y,k) = y^2 + \alpha \left(k - \frac{a}{b}\right)^2$$

is conserved along trajectories of $(*)$ if we set $\underline{\alpha = |b|}$ and $\underline{\beta = \text{sign}(b)}$. To check:

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial k}\frac{dk}{dt} \\ &= 2y\,(a - bk)y + 2\alpha\left(k - \frac{a}{b}\right)\beta y^2 \\ &= 2y^2\left(a - \alpha\beta\frac{a}{b}\right) + 2ky^2(\alpha\beta - b) \\ &\equiv 0 \qquad \text{since } \alpha\beta = |b|\,\text{sign}(b) = b \end{aligned}$$

Thus solutions to $(*)$ are confined to ellipse $y^2 + |b| \left(k - \frac{a}{b}\right)^2 = \text{constant}$, when $\beta = \text{sign}(b)$.

In general the ellipse $\phi(y,k) = y^2 + |b|\left(k - \frac{a}{b}\right)^2$ = constant will appear as in the Figure 1



The points A and B are $\left(0, \frac{a}{b} \pm \sqrt{\frac{\phi(y(0), k(0))}{|b|}}\right)$

and trajectories of (*) will follow red arrows if $b > 0$ and green arrows if $b < 0$. The points A and B are equilibrium solutions of (*). Hence, by uniqueness of solutions to (*) trajectories along elliptical arcs will converge to A as $t \to \infty$ if $b > 0$ and converge to B as $t \to \infty$ if $b < 0$.

Question: How did we 'guess' that ϕ is a conserved quantity for (*) ?

From (*)
$$\frac{dy}{dk} = \frac{dy/dt}{dk/dt} = \frac{(a-bk)y}{\beta y^2}$$

$$\Rightarrow \beta \frac{y^2}{y} dy = \beta y\, dy = (a-bk)\, dk$$

Integrate both sides

Alternative Approach

Instead of arguments as above based on a conserved quantity one can employ a function of the following form

$$V(y,k) = y^2 + |b|\,(k-\lambda)^2 \geq 0$$

$\frac{d}{dt} V(y,k)$ along trajectories of $(*)$

$$= \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial k}\dot{k}$$

$$= 2y(a-kb)y + 2|b|(k-\lambda)\beta y^2$$

$$= 2ay^2 - 2kby^2 + 2|b|k\beta y^2 - 2|b|\beta\lambda y^2$$

$$= 2ay^2 - 2kby^2 + 2kby^2 - 2b\lambda y^2$$

$$= -2y^2(\lambda b - a) \leq 0$$

if $\lambda b - a > 0$

For such λ, if $(y(0), k(0))$ is chosen in the set (sublevel set)

$$V_c = \{ (y,k) : y^2 + |b|(k-\lambda)^2 \leq c \}$$

then $(y(t), k(t))$ remain in the set V_c $\forall\, t \geq 0$. In fact one can by <u>La Salle's invariance principle</u> that $(y(t), k(t)) \longrightarrow \{k \text{ axis}\} \cap V_c$ (red line segment in Figure 2.