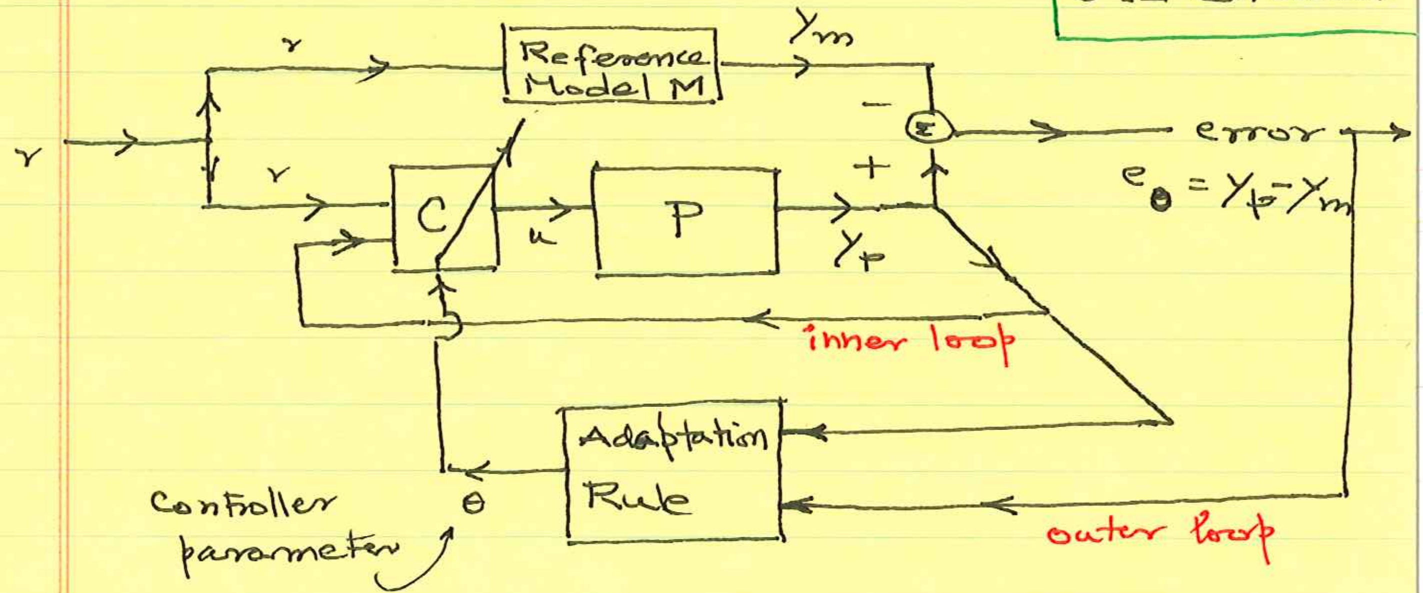
2017.09.04 ENEE 765 Lecture 2

Model Reference Adaptive System (MRAS)

P := plant
C := Controller



This is an architecture for adjusting a controller (C) for a partially known plant (P) to produce an output y_p closely matching the output y_m of a known reference model (M) when both receive the same reference input signal

MRAS example

$$P := \quad \dot{y}_p = a_p y_p + k_p u$$

$$M: \quad \dot{y}_m = a_m y_m + k_m r$$

$$C: \quad u(t) = \theta_1(t)\, r(t) + \theta_2(t)\, y_p(t)$$

$$C_{ideal}: \quad \theta_1(t) \equiv \theta_1^* = k_m / k_p \ ;$$
$$\theta_2(t) \equiv \theta_2^* = (a_m - a_p)/k_p$$

C_{ideal} can be implemented when M and P are known in which case the closed (inner) loop system is given by.

$$\dot{y}_p = a_p y_p + k_p \left(\frac{k_m}{k_p} r + \frac{a_m - a_p}{k_p} y_p \right)$$

$$= a_m y_p + k_m r$$

which matches $\underline{M}$ as desired.

To make the time-dependent controller C function similarly, even when the plant P is not known, one needs to adapt / adjust θ_1 and θ_2. The "gradient rule":

$$\begin{cases} \dot{\theta}_1 = -\gamma \, (y_p - y_m) \, r = -\gamma e_0 r \\ \dot{\theta}_2 = -\gamma \, (y_p - y_m) \, y_p = -\gamma e_0 y_p \end{cases}$$

Here γ is a constant > 0.

<u>Analysis</u>: Let $\phi_1 \triangleq \theta_1 - \theta_1^*$; $\phi_2 \triangleq \theta_2 - \theta_2^*$ denote parameter errors

[Assume that the sign of k_p is <u>known</u> and $k_p > 0$.

Then, $\dot{y}_m = a_m y_m + k_m r$

$$= (a_p + \theta_2^* k_p) y_m + \theta_1^* k_p r$$

$$= a_p y_m + k_p (\theta_1^* r + \theta_2^* y_m)$$

and $\dot{y}_p = a_p y_p + k_p (\theta_1 r + \theta_2 y_p)$.

It follows that,

$$\dot{e}_0 = \dot{y}_p - \dot{y}_m$$

$$= a_p e_0 + k_p (\theta_1 - \theta_1^*) r + k_p (\theta_2 y_p - \theta_2^* y_m)$$

$$= a_p e_0 + k_p (\theta_1 - \theta_1^*) r + k_p (\theta_2 y_p - \theta_2^* y_p + \theta_2^* y_p - \theta_2^* y_m)$$

$$= (a_p + k_p \theta_2^*) e_0 + k_p (\theta_1 - \theta_1^*) r + k_p (\theta_2 - \theta_2^*) y_p$$

$$= a_m e_0 + k_p \phi_1 r + k_p \phi_2 y_p$$

$$= a_m e_0 + k_p \phi_1 r + k_p \phi_2 (y_m + e_0)$$

Collect together all the error equations

$$\begin{cases} \dot{e}_0 = a_m e_0 + k_p \phi_1 r + k_p \phi_2 (e_0 + y_m) \\ \dot{\phi}_1 = -\gamma e_0 r \\ \dot{\phi}_2 = -\gamma e_0 (e_0 + y_m), \end{cases}$$

driven by r the reference input and y_m the output of the reference model.

Three special cases:

(i) k_p is known. So we can set $\theta_1(t) \equiv \theta_1^* = k_m/k_p$. Drop $\dot{\phi}_1 = 0$ equation, and let

$$\phi \triangleq \phi_2$$

$$\Rightarrow \begin{cases} \dot{e}_0 = a_m e_0 + k_p \phi (e_0 + y_m) \\ \dot{\phi} = -\gamma e_0 (e_0 + y_m) \end{cases}$$

(ii) ~~$a_m < 0$~~ $r(t) \equiv 0$; $y_m(0) = 0 \Rightarrow y_m(t) \equiv 0$

$$\begin{cases} \dot{e}_0 = a_m e_0 + k_p \phi e_0 \\ \quad = (a_m + k_p \phi) e_0 \\ \dot{\phi} = -\gamma e_0^2 \end{cases}$$

As in lecture 1

$$\Psi(e_0, \phi) = \gamma e_0^2 + k_p \left(\phi + \frac{a_m}{k_p}\right)^2$$

is conserved. Check,

$$\frac{d\Psi}{dt} = \frac{\partial \Psi}{\partial e_0}\dot{e}_0 + \frac{\partial \Psi}{\partial \phi}\dot{\phi}$$

$$= 2\gamma e_0 (a_m + k_p\phi) e_0 + 2k_p \left(\phi + \frac{a_m}{k_p}\right)(-\gamma e_0^2)$$

$$= \cancel{2\gamma a_m e_0^2} + \cancel{2k_p\phi e_0^2\gamma} - \cancel{2k_p\gamma\phi e_0^2} - \cancel{2k_p\gamma e_0^2 \frac{a_m}{k_p}}$$

$$\equiv 0$$

Since γ is chosen > 0 and k_p is assumed to be > 0,

$$\Psi(e_0, \phi) = c^2$$

defines ellipses for $c \neq 0$ and the trajectories $(e_0(t), \phi(t))$ are confined to such ellipses.
In fact $e_0(t) \to 0$ as $t \to \infty$

but notice $\phi(t) \to \phi_\infty = -\frac{a_m}{k_p} \pm \sqrt{\frac{\Psi(e_0(0), \phi(0))}{k_p}}$

which sign is correct?

(iii) Suppose a_p is known $= a_m$, but k_p is unknown except that we know $k_p > 0$.

Then $\theta_2^* = 0$, so it makes sense to set $\theta_2(t) \equiv \theta_2^* = 0 \Rightarrow \phi_2 \equiv 0$. Drop ϕ_2 equation, let $\underline{\phi \triangleq \phi_1}$ to obtain

$$(*) \quad \begin{cases} \dot{e}_0 = a_m e_0 + k_p \phi r \\ \dot{\phi} = -\gamma e_0 r \end{cases}$$

~~Any~~ Point ~~$(0, \phi)$~~ $(0,0)$ is ~~an~~ unique equilibrium point of the system.

Let $V(e_0, \phi) = \gamma e_0^2 + k_p \phi^2$. Along trajectories of $(*)$

$$\frac{dV}{dt} = \frac{\partial V}{\partial e_0} \dot{e}_0 + \frac{\partial V}{\partial \phi} \dot{\phi}$$

$$= 2\gamma e_0 (a_m e_0 + k_p \phi r)$$

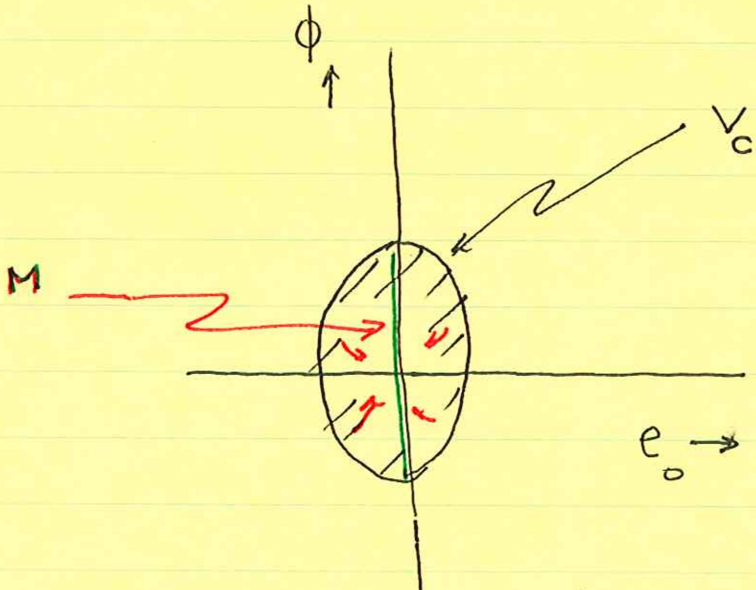
$$+ 2 k_p \phi (-\gamma e_0 r)$$

$$= 2 \gamma a_m e_0^2$$

Suppose $a_m < 0$. Then $\frac{dV}{dt} \le 0$ ~~along~~ trajectories of $(*)$. On the other hand

$$V_c = \{ (e_0, \phi) : V(e_0, \phi) \le c \}$$

is closed and bounded set (ellipsoidal region). Since $\dot{V} \le 0$, V_c is a positively invariant set for $(*)$, i.e. trajectories starting inside V_c stay inside and in fact can be shown to converge to the green line segment marked by the letter M.



we will visit this later

The argument in favor of this is complicated by the fact that ~~the~~ the system $(*)$ is non-autonomous (has time-dependent dynamics) since $r(t)$ is a function of time. Only the origin is an equilibrium.

If $r(t) \equiv$ constant $= 0$, then ϕ does not change and $\dot{e}_0 = a_m e_0 \Rightarrow e_0(t) \to 0$

as $t \to \infty$ under the assumption $a_m < 0$.
If $r(t) \equiv$ constant $= c \neq 0$, then

$$\ddot{e}_0 + (-a_m)\dot{e}_0 + (\gamma k_p c^2) e_0 = 0$$

and,

$$\ddot{\phi} + (-a_m)\dot{\phi} + (\gamma k_p c^2)\phi = 0 .$$

Hence e_0 and ϕ both undergo oscillatory decay to 0.