Lectures on Nonlinear Control Systems

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Lecture 1

Introduction to Nonlinearity

In this course we will discuss nonlinear control theory from the point of view of understanding the main principles and techniques that shed light on qualitative properties of such systems. We will address:

(i) Controllability - When does there exist a control that drives the system from an initial state to a prescribed target state?

(ii) Observability - Can you infer state from observations of an output signal?

(iii) Special solutions - equilibria, periodic orbits, and bifurcations with respect to parameter variation

(iv) Stability of solutions - a central topic

Further, we will discuss how this understanding leads to approaches for design. Our techniques will include algebraic, geometric, and analytic methods in the study of differential equations.

Nonlinearity arises in a number of ways:

(i) If the state space is not a vector space. For instance, in the control of a magnetic moment using external fields, the state space is a sphere

$$\left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = r^2\right\}$$

(ii) If the equations of motion are nonlinear. For instance, the pendulum

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = u$$

where $u =$ torque applied at the pivot.

(iii) If actuators (or sensors) are subject to nonlinear constitutive relations - e.g. hysteresis.

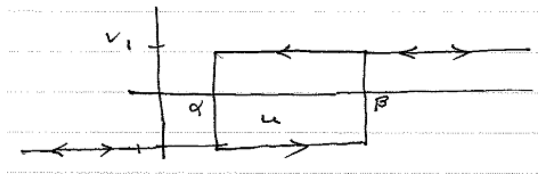


Figure 1.1: Hysteresis

Increasing u from $-\infty$ to β leaves v constant $= -1$ until a jump occurs for $u = \beta$ and thereafter v remains at 1 for further *increase* in u.

Decreasing u from $+\infty$ to α leaves v constant $= +1$ until a jump occurs $u = \alpha$ and thereafter v remains at -1 for further *decrease* in u.

Magnetic recording processes depend on hysteresis. Other applications of hysteresis arise in actuators incorporating deformable materials.

Example 1.1. Consider the controlled pendulum in the adjoining figure.

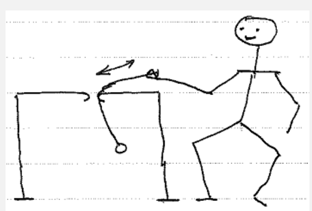


Figure 1.2: Controlled pendulum

The pendulum is suspended on a string fed through a hole on a table top and controlled by an investigator. The investigator controls the length of the pendulum (possibly periodically). The interaction of the pendulum with the table introduces a *frictional torque*.

Approximating $\sin(\theta)$ by θ (small oscillation assumption) and letting $x = \theta$, we obtain the model (with damping constant $b > 0$):

$$\ddot{x} + v(t)x = -b\dot{x}$$

where $v(t) = \frac{g}{l(t)}$ is interpreted as a control that depends on the time function used by the investigator. We thus have a state space model:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + v(t)\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\longleftrightarrow \qquad \dot{z} = Az + vBz$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

Here the control enters multiplicatively. If v = constant, these dynamics describe the free, damped oscillation of a pendulum with natural frequency $= \sqrt{v}$.

Example 1.2. Consider the *unicycle* seen from above in the adjoining figure.

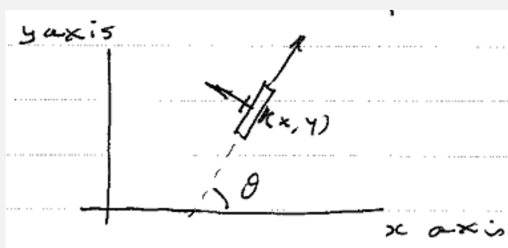


Figure 1.3: Unicycle in the plane

Forward speed (by pedaling) is u. Steering rate is ω. It is then easy to show that

$$\begin{aligned} \dot{x} &= u\cos(\theta) \\ \dot{y} &= u\sin(\theta) \\ \dot{\theta} &= \omega \end{aligned}$$

We can repackage this as,

$$\dot{g} = g\xi$$

where

$$g = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \xi = \omega\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrices of the form g above constitute a matrix (Lie) group with multiplication:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & \hat{x} \\ \sin(\phi) & \cos(\phi) & \hat{y} \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & x+\hat{x}\cos(\theta)-\hat{y}\sin(\theta) \\ \sin(\theta+\phi) & \cos(\theta+\phi) & y+\hat{x}\sin(\theta)+\hat{y}\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse is given by,

$$g^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & -x\cos(\theta)-y\sin(\theta) \\ \sin(-\theta) & \cos(-\theta) & x\sin(\theta)-y\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

The collection of all such g matrices constitutues the rigid motion group in the plane SE(2). Formally,

$$\mathbb{SE}(n) = \left\{ \begin{bmatrix} A & b \\ \underline{0} & 1 \end{bmatrix} : A^T A = I_n, b \in \mathbb{R}^n, \det(A) = 1, \text{ and } \underline{0} = \text{row vector of n zeros} \right\}$$

The block $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for $n = 2$ is just a planar, counterclockwise rotation by θ.

Thus motion of a unicycle in the plane gives a curve in $\mathbb{SE}(2)$ with two controls ω and u. If the controls are set to zero, then there is no motion, i. e. we have a drift-free system.

$\mathbb{SE}(2)$ is <u>not</u> a vector space. It is an example of a smooth manifold.

Lecture 2

Frenet-Serret Equations: Control on a Lie group

2.1. Frenet-Serret Frame

Consider a C^3 curve in $\mathbb{R}^3$, $t \mapsto \gamma(t)$ starting at $\gamma(t_0) = \gamma_0$.

Let $s(t) = \int_{t_0}^{t} (\frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt})^{1/2} dt$ denote the length of the curve γ from t_0 to t. The dot product is the Euclidean inner product.

Then, speed $\frac{ds}{dt} = \|\dot{\gamma}(t)\| = (\dot{\gamma}(t) \cdot \dot{\gamma}(t))^{1/2}$.

Hypothesis 1. $\dot{\gamma}(t) \neq 0$ *for any* $t \geq t_0$ *(regular curve). Then* $s(t)$ *is strict monotonic function of* t *and can be inverted in principle to obtain* $t = t(s)$*. Note,* $t_0 = t(0)$*. Thus the curve can be re-parameterized in terms of* s *by expressing* $\gamma = \gamma(t) = \gamma(t(s))$*.*

Definition 2.1. *We call the above re-parameterization, the* <u>*arc-length parameterization*</u>*. We can write* <u>*tangent*</u>

$$T(s) \triangleq \frac{d\gamma}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{d\gamma}{dt} / \frac{ds}{dt}.$$

△

Then,

$$\|T(s)\| = \frac{\left\|\frac{d\gamma}{dt}\right\|}{\left|\frac{ds}{dt}\right|} = \frac{\frac{ds}{dt}}{\frac{ds}{dt}} = 1$$

for all $s \geq 0$.

Thus, in the arc-length parameterization, the curve γ has unit speed. So, we also refer to the arc length parameterization as the unit speed parameterization.

Remark 2.1. Changing the (laboratory) coordinate system into a new one by rotation and translation, the original curve γ becomes a new curve $\tilde{\gamma}$.

$$\tilde{\gamma}(t) = P\gamma(t) + b$$

where $P \in \mathbb{SO}(3)$ and $b \in \mathbb{R}^3$. Since $\dot{\tilde{\gamma}}(t) = P\dot{\gamma}(t)$, it follows that the arc-length,

$$\tilde{s}(t) = \int_0^t \left\| \frac{d\tilde{\gamma}}{dt} \right\| dt = \int_0^t \left\| \frac{d\tilde{\gamma}}{dt} \right\| dt = \int_0^t \left\| \frac{d\gamma}{dt} \right\| dt = s(t)$$

i.e. arc-length is invariant under $\mathbb{SE}(3)$ action. We seek other invariants. $\triangle$

Definition 2.2. *Natural curvature* $\kappa(s) \triangleq \left\| \frac{dT}{ds} \right\| \geq 0.$ $\triangle$

Natural curvature is also an invariant under $\mathbb{SE}(3)$ action $\gamma \mapsto P\gamma + b$.

Property 1. $\kappa(s) \equiv 0$ on an interval of definition of a curve if and only if $\gamma(s)$ is a straight line on that interval. $\triangle$

Property proof 1.

$$\begin{aligned}
(\Rightarrow) \quad \kappa(S) \equiv 0 \quad &\Leftrightarrow \quad \left\| \frac{dT}{ds} \right\| \equiv 0 \quad \text{on an interval} \\
&\Leftrightarrow \frac{dT}{ds} \equiv 0 \quad \text{on an interval} \\
&\Rightarrow T(s) \equiv \text{constant} = \mathbf{c} \\
&\Leftrightarrow \frac{d\gamma}{ds} = \mathbf{c} \\
&\Leftrightarrow \gamma(s) = \gamma(0) + s\mathbf{c} \qquad \text{(straight line)} \\
(\Leftarrow) \quad &\text{Trace backward the above steps.}
\end{aligned}$$

$\blacksquare$

Remark 2.2. Note that

$$T(s) \cdot T(s) \equiv 1.$$

Differentiate to obtain,

$$T'(s) \cdot T(s) \equiv 0,$$

where $'$ denotes $\frac{d}{ds}$. $\triangle$

Definition 2.3. *(Normal, Binormal, and Frenet-Serret Frame).*
If $\kappa(s_1) \neq 0$ for a particular s_1 then we can define the unit normal vector.

$$N(s_1) = \frac{T'(s_1)}{\kappa(s_1)}$$

By continuity, such a normal is defined on a neighborhood of s_1. On that neighborhood, one defines the unit binormal vector,

$$B(s) = T(s) \times N(s)$$

and thus obtains the orthonormal triad $\{T(s), N(s), B(s)\}$. We call this the Frenet-Serret frame of the curve. △

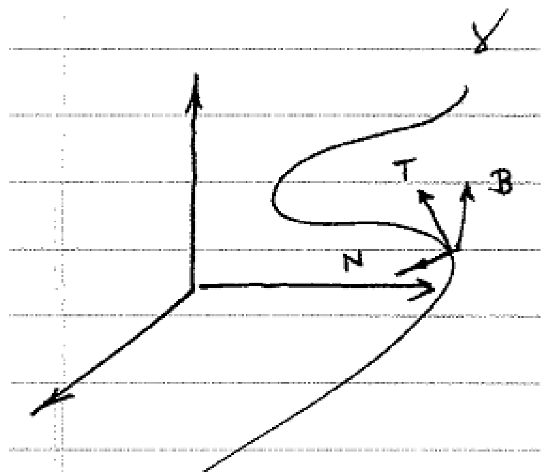


Figure 2.1: Frenet-Serret Frame

2.2. Frenet-Serret Equations

Recall that this construction works only on a neighborhood of s_1 where $\kappa(s_1) \neq 0$, to avoid division by zero in the definition of N. To make this work for all s, we need an additional hypothesis.

Hypothesis 2. *(Non-degeneracy).*

$$\kappa(s) \neq 0 \qquad \forall s$$

The non-degeneracy hypothesis holds generically. Under this hypothesis, one can derive a set of differential equations to evolve the triad $\{T(s), N(s), B(s)\}$.

Let $F \triangleq [F_1(s) \quad F_2(s) \quad F_3(s)] \triangleq [T(s) \quad N(s) \quad B(s)]$. Clearly, $F^T F \equiv (I)$ and $\det(F) = +1$, since the triad $\{T(s), N(s), B(s)\}$ is right-handed. Thus, $s \mapsto F(s)$ defines a curve in $\mathbb{SO}(3)$. Further, we will see that F is generated by a skew symmetric (s-dependent) matrix $\hat{\Omega}$:

$$\frac{F(s)}{ds} = F(s)\hat{\Omega}$$

where $\hat{\Omega} + \hat{\Omega}^T \equiv 0$. The $\hat{\cdot}$ operator here represents an operator which forms a cross-product equivalent matrix from a given vector argument.

The structure of $\hat{\Omega}$ is easy to work out. For s skew-aymmetric matrix, we can write,

$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} \frac{dF_1}{ds} = \frac{dT}{ds} &= F(s) \cdot 1^{\text{st}} \text{ column of } \hat{\Omega} \\ &= T(s) \cdot 0 + N(s)\Omega_3(s) + B(s)(-\Omega_3) \\ &= N(s)\kappa(s) \qquad \text{(by definition of N)} \end{aligned}$$

$$\Rightarrow \Omega_3 = \kappa \quad \text{and} \quad \Omega_2 \equiv 0.$$

Similarly,

$$\begin{aligned} \frac{dF_2}{ds} = \frac{dN}{ds} &= F(s) \cdot 2^{\text{nd}} \text{ column of } \hat{\Omega} \\ &= T(s)(-\Omega_3) + B(s)(\Omega_1) \\ &= -\kappa T(s) + \tau B(s) \end{aligned}$$

where we define $\tau(s) \triangleq \Omega_1(s)$, (torsion). Also,

$$\begin{aligned} \frac{dF_2}{ds} = \frac{dN}{ds} &= F(s) \cdot 3^{\text{rd}} \text{ column of } \hat{\Omega} \\ &= -\tau(s)N(s) \end{aligned}$$

The last equation also tells us,

$$\tau(s) = -\frac{dB}{ds} \cdot N(s)$$

We can take this to be the definition of torsion.

Thus, we have the Frenet-Serret equations

$$\frac{d}{ds} \begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} = \begin{bmatrix} T(s) & N(s) & B(s) \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}$$

Given a program of curvature, $\kappa(s)$, and torsion, $\tau(s)$, we can integrate the above system of equations starting from the initial frame and compute the curve γ by,

$$\gamma(s) = \gamma(0) + \int_0^s T(\sigma)d\sigma$$

Property 2. A curve is planar if and only if $\tau(s) \equiv 0$. $\triangle$

Property proof 2. Recall that we say γ is planar if there is a fixed non-zero vector μ such that $\mu \cdot \gamma(s) \equiv$ constant

$$\tau(s) \equiv 0 \Leftrightarrow \frac{dB}{ds} \equiv 0 \Leftrightarrow B(s) \equiv \text{constant}$$

$$\begin{aligned}
(\Rightarrow) \quad & \text{Suppose } B(s) \equiv 0 \quad \text{(a constant vector } \mu\text{)} \\
\Rightarrow \quad & B(\sigma) \cdot T(\sigma) = B(s) \cdot T(\sigma) = \mu \cdot T(\sigma) \equiv 0 \\
\Rightarrow \quad & \mu \cdot \gamma(s) = \mu \cdot \gamma(0) + \int_0^s \mu \cdot T(\sigma) d\sigma \\
& \qquad = \mu \cdot \gamma(0) = \text{constant} \Rightarrow \text{planar}
\end{aligned}$$

$$\begin{aligned}
(\Leftarrow) \quad & \text{Suppose } \mu \cdot \gamma(s) \equiv \text{constant, } \mu \neq 0 \text{ (planar),} \\
\Rightarrow \quad & \mu \cdot \gamma'(s) = \mu \cdot T(s) \equiv 0 \\
\Rightarrow \quad & \mu \cdot T'(s) = \kappa(s)\mu \cdot N(s) \equiv 0 \\
& \text{Since} \quad \kappa(s) \neq 0 \quad \text{(nondegeneracy),} \\
& \quad \mu \cdot N(s) \equiv 0 \\
\Rightarrow \quad & 0 \equiv \mu \cdot N'(s) = -\kappa(s)\mu \cdot T(s) + \tau(s)\mu \cdot B(s) \\
& \quad = 0 + \tau(s) \cdot (\mu \cdot B(s))
\end{aligned}$$

Since $\mu \cdot T(s) \equiv 0$ and $\mu \cdot N(s) \equiv 0$, it is necessary that $\mu \cdot B(s) \neq 0$ for any s. Otherwise, the constant vector

$$\begin{aligned}
\mu &= (\mu \cdot T(s))T(s) + (\mu \cdot N(s))N(s) + (\mu \cdot B(s))B(s) \\
&= 0
\end{aligned}$$

But $\tau(s) \cdot (\mu \cdot B(s)) = 0$, hence $\tau(s) \equiv 0$. ■

2.3. Kinematics of particles in $\mathbb{R}^3$

Suppose a particle in $\mathbb{R}^3$ traces a trajectory $\gamma(t)$ where $t =$ time. Let $s(t) =$ be the arc length along the trajectory traversed in time t,

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt} \right\| \cdot dt.$$

Let $\nu = \frac{ds}{dt}$ denote the speed.

Then,

$$\begin{aligned}\mathbf{v}(t) &= \text{velocity}\\ &= \frac{d\gamma}{dt}\\ &= \frac{d\gamma}{ds}\frac{ds}{dt}\\ &= T(s)\frac{ds}{dt}\\ &= \nu(s)T(s)\end{aligned}$$

Let $g(s)$ provide the location and orientation of the Frenet-Serret frame, packaged in a convenient manner. That is, let

$$g(s) = \begin{bmatrix} F(s) & \gamma(s) \\ 0 & 1 \end{bmatrix} \in \mathbb{SE}(3).$$

Then

$$\begin{aligned}\frac{dg}{ds} &= \begin{bmatrix} F\hat{\Omega}(s) & \frac{d\gamma}{ds} \\ 0 & 0 \end{bmatrix}\\ &= g(s)\cdot\begin{bmatrix} \hat{\Omega}(s) & e_1 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{2.1}$$

where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Equation (2.1) is a control system on a Lie group, controlled by the curvature and torsion. It is very interesting to consider optimal control problems of the form,

$$\begin{aligned} & \min \int_0^L (\kappa^2(s) + \gamma^2(s))ds \\ \text{subject to} \quad & \kappa(s) > 0, \quad s \in [0, L] \\ & g(0) = \mathbb{I}_{4\times 4} \\ & g(L) = g_1 \quad \text{prescribed} \\ \text{and} \quad & \frac{dg}{ds} = g\left[\begin{array}{c|c} \hat{\Omega} & e_1 \\ \hline 0 & 0 \end{array}\right]\end{aligned}$$

We can also alternately express everything in the original non-unit speed parameterization, t.

$$\frac{dg}{dt} = g \left[\begin{array}{c|c} \nu\hat{\Omega} & \nu e_1 \\ \hline 0 & 0 \end{array} \right] \tag{2.2}$$

where $\nu =$ speed (is a function of t).

Lecture 3

Lie Groups and Lie Algebras

Definition 3.1. *A set S together with an operation denoted by $(\cdot) : S \times S \to S$, is a group if the following axioms hold:*

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in S$

(ii) there is an element $e \in S$ such that $a = e \cdot a = a \cdot e \quad \forall a \in S$. ($e$ is the identity element; if an identity element exists, it is unique).

(iii) for each $a \in S$ there is an element b such that $a \cdot b = b \cdot a = e$. It can be shown that '$b$' is uniquely determined by a and we denote 'b' as a^{-1}.

We call the pair $G = (S, \cdot)$ a group. $\triangle$

Example 3.1. $G = (\mathrm{Gl}(n, \mathbb{R}), \cdot)$ where $\mathrm{Gl}(n, \mathbb{R})$ denotes the set of all $n \times n$ nonsingular matrices with matrix multiplication as the operation that completes the group structure. This is the general linear group.

Definition 3.2. *A subset $Q \subset S$ where $G = (S, \cdot)$ is a group can also inherit the group structure from G, provided,*

(i) $a, b \in Q \implies a \cdot b \in Q$

(ii) e the identity element in S is also in Q

(iii) $a \in Q \implies a^{-1} \in Q$

In this case, we call $\tilde{G} = (Q, \cdot)$ a subgroup of $G = (S, \cdot)$. △

Example 3.2. O$(n, \mathbb{R})$ the set of all $n \times n$ real orthogonal matrices is a subgroup of Gl$(n, \mathbb{R})$. Note, for shorthand we have omitted the group operation when referring to the group (O$(n, \mathbb{R}), \cdot)$ as simply O$(n, \mathbb{R})$. We have made a similar abbreviation for Gl$(n, \mathbb{R})$. Subsequently, for matrix groups, the matrix multiplication operation will be implied.

Example 3.3. Let SO$(n, \mathbb{R}) = \{M \in$ O$(n, \mathbb{R}) : det(M) = 1\}$. Then SO$(n, \mathbb{R})$ is a subgroup of O$(n, \mathbb{R})$. It is the special orthogonal group.

Definition 3.3. *A group is abelian if $a \cdot b = b \cdot a \quad \forall a, b \in G$.* △

Example 3.4. $G = (\mathbb{R}, +)$, $G = (\mathbb{R}^n, +)$, $G = (\text{Mat}(n), +)$, $G =$ SO$(2, \mathbb{R})$ are all abelian groups. Gl$(n, \mathbb{R})$ for $n \geq 2$ is not abelian.

Definition 3.4. *Given two groups $G_1 = (S_1, \cdot_1)$ and $G_2 = (S_2, \cdot_2)$, we define the direct product of these two groups to be $G = (S, \cdot)$, where $S = S_1 \times S_2$ (the cartesian product of the sets) and $(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2)$ for $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.* △

Direct products give us a way to define new groups out of building blocks of other groups.

Example 3.5. Let $G_1 =$ SO$(2, \mathbb{R})$ and $G_2 = (\mathbb{R}^2, +)$, then $G =$ SO$(2, \mathbb{R}) \times \mathbb{R}^2$ with a multiplication rule given by

$$\left(\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) =$$
$$\left(\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}, \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \right)$$

Contrast this group with the group SE$(n, \mathbb{R})$, encountered in our previous discussion of the unicycle model. These groups are NOT the same, since the multiplication rules are different. $G =$ SO$(2, \mathbb{R}) \times \mathbb{R}^2$ derives its multiplication rule from combining the multiplication in SO$(2, \mathbb{R})$ and the vector addition in $\mathbb{R}^2$. In contrast, the semi-direct product SE$(2, \mathbb{R})$ derives its multiplication rule from matrix multiplication as a subgroup of Gl$(3, \mathbb{R})$.

The matrix groups encountered so far are all subgroups of Gl$(n, \mathbb{R})$ which in turn is an open subset (because of the condition $\det(X) \neq 0$) of Mat$(n, \mathbb{R})$ the set of all $n \times n$ matrices over the reals. Mat$(n, \mathbb{R})$ is clearly a vector space of dimension n^2 and can be equipped with metrics (from norms) in a number of different ways. For instance, a ball $B_{M_o}(\epsilon)$ of radius $\epsilon > 0$ centered at M_o in Mat$(n, \mathbb{R})$ can be defined to be

$$B_{M_o}(\epsilon) = \{M \in \text{Mat}(n, \mathbb{R}) : (tr((M - M_o)^T(M - M_o)))^{1/2} < \epsilon\}$$

This is the open Euclidean ball in Mat$(n, \mathbb{R})$ defining what is known as the usual topology. Gl$(n, \mathbb{R})$ inherits this topology by the following definition.

Definition 3.5. *$U \subset$ Gl$(n, \mathbb{R})$ is an open set in Gl$(n, \mathbb{R})$ if and only if $U =$ Gl$(n, \mathbb{R}) \cap V$ where V is an open subset of Mat$(n, \mathbb{R})$. V is an open subset of Mat$(n, \mathbb{R})$ if and only if for each $M_o \in V$ there is an $\epsilon = \epsilon(M_o) > 0$ such that $B_{M_o}(\epsilon) \subset V$ is a strict subset of V.* △

Figure 3.1 depicts these relationships.

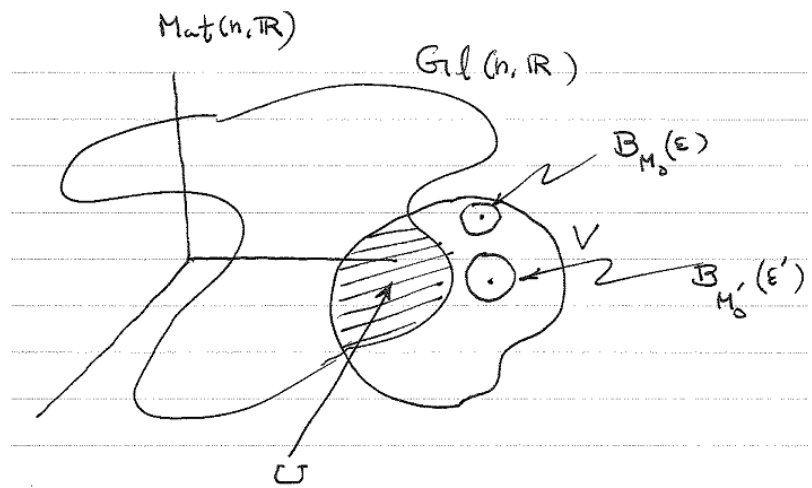


Figure 3.1: Graphic of subset relationships.

Observe that the definition of SO$(n, \mathbb{R})$ as a subgroup of Gl$(n, \mathbb{R})$ allows us to introduce the subspace topology on SO$(n, \mathbb{R})$: $V \subset$ SO$(n, \mathbb{R})$ is open if and only if $V =$ SO$(n, \mathbb{R}) \cap U$ where $U \subset$ Gl$(n, \mathbb{R})$ is open. All subgroups of Gl$(n, \mathbb{R})$ inherit a topology in this way.

One can actually show more: Gl$(n, \mathbb{R})$ can be given the structure of a manifold (i.e. open sets can be used to cover Gl$(n, \mathbb{R})$ in such a way as to yield coordinate charts $\{(U_\alpha, \phi_\alpha) : U_\alpha \subset$ Gl$(n, \mathbb{R})$ open and $\phi_\alpha : U_\alpha \to \mathbb{R}^k$ smooth map, invertible and with smooth inverse, $\alpha \in A =$ index set$\}$). This is a starting point for thinking of Gl$(n, \mathbb{R})$ as a smooth manifold.

The classical groups (and their subgroups) are of great importance for physical problems. We list them over the reals $\mathbb{R}$ and complexes $\mathbb{C}$.

Classical groups over $\mathbb{R}$:

- General linear group Gl$(n,\mathbb{R}) = \{X : X$ is an $n\times n$ matrix with $\det(X) \neq 0\}$
- Special linear group Sl$(n,\mathbb{R}) = \{X : X \in$ Gl$(n,\mathbb{R}), \det(x) = 1\}$
- Orthogonal group O$(n,\mathbb{R}) = \{X : X \in$ Gl$(n,\mathbb{R}), X^T X = I_n\}$
- Special orthogonal group SO$(n,\mathbb{R}) =$ O$(n,\mathbb{R}) \cap$ Sl$(n,\mathbb{R})$
- Symplectic group Sp$(2n,\mathbb{R}) = \{X : X \in$ Gl$(n,\mathbb{R}), X^T J X = J\}$
- Pseudo-orthogonal group O$(p,q,\mathbb{R}) = \{X : X \in$ Gl$(p+q,\mathbb{R}), X^T\Sigma_{p,q}X = \Sigma_{p,q}\}$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $\Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.

Classical groups over $\mathbb{C}$:

One replaces the transpose operation by the Hermitian transpose $(\cdot^*)$ (complex conjugate transpose). In particular, we refer to two following groups:

- Unitary group U$(n,\mathbb{C}) = \{X : X \in$ Gl$(n,\mathbb{C}), X^* X = I_n\}$
- Special unitary group SU$(n,\mathbb{C}) =$ Sl$(n,\mathbb{C}) \cap$ U$(n,\mathbb{C})$.

Since the groups above are embedded in the space Mat$(n,\mathbb{R})$ (or Mat$(n,\mathbb{C})$), it makes sense to speak of a <u>curve</u> in a classical group that is continuously differentiable with respect to its parameter. Thus, consider

$$t \mapsto \Phi(t) \in \text{SO}(n,\mathbb{R})$$

a differentiable curve for $t \in [0,T]$.
Then,

$$\Phi^T(t)\Phi(t) = I_n \quad \forall t \in [0,T]$$

Differentiating both sides, we get,

$$\dot{\Phi}^T(t)\Phi(t) + \Phi^T(t)\dot{\Phi}(t) \equiv 0$$

$$\implies (\Phi^T(t)\dot{\Phi}(t))^T + (\Phi^T(t)\dot{\Phi}(t)) \equiv 0$$

$$\implies \Phi^T(t)\dot{\Phi}(t) = \xi(t) \quad n \times n \text{ skew-symmetric matrix-valued function of } t.$$

Equivalently,

$$\dot{\Phi}(t) = \Phi(t)\xi(t) \tag{3.1}$$

since $(\Phi^T(t))^{-1} = (\Phi^{-1}(t))^{-1} = \Phi(t)$.

Thus, to each smooth curve in SO(n), one can associate a smooth curve in so(n), the space of $n \times n$ skew-symmetric matrices. Conversely, given any continuous curve $\xi(t)$ in so(n), and $\Phi(0) \in$ SO(n), one can produce (by integration) an unique curve $\Phi(t)$ in SO(n). The proof of this converse is not so obvious, but we can see it easily in a special case: $\xi(t) \equiv \xi$ a constant skew-symmetric matrix. In that case, by the theory of differential equations,

$$\Phi(t) = \Phi(0)e^{t\xi}.$$

Hence,

$$\begin{aligned}
\Phi(t)\Phi^T(t) &= \Phi(0)e^{t\xi}e^{t\xi^T}\Phi^T(0) \\
&= \Phi(0)e^{t\xi}e^{-t\xi}\Phi^T(0) \\
&= \Phi(0)e^{t(\xi-\xi)}\Phi^T(0) \\
&= I_n \qquad \forall t \in [0,T].
\end{aligned}$$

To prove the converse in general for a time dependent ξ, one needs a representation of the solution to the differential equation (3.1). See the Wei-Norman (1964) paper.

In a similar way, consider $t \mapsto \Phi(t)$ a smooth curve in Sp$(2n, \mathbb{R})$. Then

$$\Phi^T(t)J\Phi(t) \equiv J$$

Differentiating both sides, we get

$$\begin{aligned}
\dot{\Phi}^T(t)J\Phi(t) + \Phi^T(t)J\dot{\Phi}(t) &\equiv 0 \\
-\dot{\Phi}^T(t)J^T\Phi(t) + \Phi^T(t)J\dot{\Phi}(t) &\equiv 0 \qquad (\text{since } J = -J^T) \\
-(\Phi^T(t)J\dot{\Phi}(t))^T + \Phi^T(t)J\dot{\Phi}(t) &\equiv 0
\end{aligned}$$

Thus $\tilde{\xi}(t) \triangleq \Phi^T J\dot{\Phi}$ is a symmetric matrix-valued function. Note that

$$\begin{aligned}
(J\tilde{\xi})^T J + J(J\tilde{\xi}) &= \tilde{\xi}^T J^T J + JJ\tilde{\xi} \\
&= \tilde{\xi}^T - \tilde{\xi} \\
&= 0
\end{aligned}$$

Hence $J\tilde{\xi} : [0,T] \to$ sp$(2n)$, where sp$(2n) = \{X : X^T J + JX = 0\}$. We call sp$(2n)$ the space of hamiltonian (or infinitesimally symplectic) matrices. It is clearly a vector space, and since

$$
\begin{aligned}
& \Phi^T J \dot{\Phi} = \tilde{\xi} \\
\iff\ & \dot{\Phi} = J^{-1}(\Phi^T)^{-1}\tilde{\xi} \\
& \quad = -\Phi J \tilde{\xi} && \text{(since } \Phi^T J \Phi = J \text{ and } J^{-1} = J^T = -J\text{)} \\
& \quad \triangleq \Phi\xi && \text{(where } \xi \triangleq -J\tilde{\xi}\text{)}
\end{aligned}
$$

Hence, we again have the matrix differential equation

$$\dot{\Phi} = \Phi\xi.$$

It follows that sp$(2n)$ plays the same role for Sp$(2n)$ as does so(n) for SO(n). In particular, if $\xi(t) \equiv \xi$ constant $\in$ sp$(2n)$, then

$$t \mapsto \exp(t\xi) \in \text{Sp}(2n)$$

$\forall t \in \mathbb{R}$. Also note that the constraint derived for $\tilde{\xi}$ to be symmetric can be put in terms of ξ. This constraint is used to defined the space sp$(2n, \mathbb{R})$.

The above construction is applicable to all the classical groups.

Definition 3.6. *Corresponding vector spaces:*

- *gl*$(n, \mathbb{R}) = \{$*all* $n \times n$ *matrices*$\}$
- *sl*$(n, \mathbb{R}) = \{X : X \in gl(n, \mathbb{R}), tr(X) = 0)\}$
- *so*$(n, \mathbb{R}) = \{X : X \in gl(n, \mathbb{R}), X^T + X = 0\}$
- *sp*$(2n, \mathbb{R}) = \{X : X \in gl(2n, \mathbb{R}), X^T J + JX = 0\}$
- *so*$(p, q, \mathbb{R}) = \{X : X \in gl(p+q, \mathbb{R}), X^T\Sigma_{p,q} + \Sigma_{p,q}X = 0\}$

These vector spaces each have the important property that,

$$X \in V \implies \exp(X) \in G$$

where V has been used as a generic symbol to represent one of the vector spaces just defined, and G is used to represent the corresponding classical group. We note that the exponential maps take values in appropriate classical groups. However, in general, it is <u>not onto</u>. For example, there does NOT exist a real matrix X such that

$$\exp(X) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \in Gl(2, \mathbb{R})$$

$\triangle$

Since,

$$\frac{d}{dt}\exp(tX) = \exp(tX)X$$

and $\exp(0 \cdot X) = I$ at $t = 0$, we have that $\frac{d}{dt}\exp(tX) = X$ at $t = 0$. Therefore, we can interpret gl(n), sl(n), so(n), sp$(2n)$, and so(p, q) as the spaces where velocities of curves passing through identity in corresponding classical groups live. These vector spaces also carry another, algebraic structure.

Definition 3.7. *A vector space V, together with an operation (Lie bracket)*

$$\begin{aligned}[\cdot,\cdot] : V \times V &\to V \\ (a, b) &\mapsto [a, b]\end{aligned}$$

is said to constitute a Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$ is the operation above satisfies the axioms:

(i) $[a, b] = -[b, a]$

(ii) $[\alpha a+\beta b, c] = \alpha[a, c]+\beta[b, c]$ *where* $\alpha, \beta \in \mathbb{F}$, *the underlying field of scalars for* V.

(iii) (The Jacobi identity) $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$

△

Defining $[X, Y] = XY - YX$ to be the matrix commutation for matrices X, Y causes each of the spaces gl(n), sl(n), so(n), sp$(2n)$, and so(p, q) to be a Lie algebra. These are the classical Lie algebras.

Definition 3.8. *For any subgroup, $G \subseteq Gl(n)$, we define the associated Lie algebra to be the vector space $\mathfrak{g} = \{X \in gl(n) : \exp(tX) \in G \quad \forall t \in \mathbb{R}\}$. See Theorem 17 in R. Howe's Very Basic Lie Theory.* △

Definition 3.9. *Given a set of matrices $\{A_1, A_2, \ldots, A_k\}$ of size $n \times n$, we define*

$$\mathfrak{g} = L.A.\{A_1, A_2, \ldots, A_k\}$$

to be the smallest Lie algebra generated by $A_1, A_2, \ldots, A_k$ if

(i) the underlying vector space contains the linear span of $\{A_1, A_2, \ldots, A_k\}$

(ii) is closed under the Lie bracket

(iii) there is no lower dimensional space satisfying (i) and (ii).

$\triangle$

The dimension of the Lie algebra is the dimension of the underlying vector space. A Lie algebra of $n \times n$ matrices, being necessarily a subspace of $\mathrm{gl}(n)$, has dimension at most $= n^2$.
Let $\mathfrak{g} = (V, [\cdot, \cdot])$ and let $\xi_1, \xi_2, \ldots, \xi_m$ constitute a basis for V. Then, $[\xi_i, \xi_j]$ being an element of V, can be uniquely written as a linear combination of $\xi_1, \xi_2, \ldots, \xi_m$,

$$[\xi_i, \xi_j] = \sum_{k=1}^{m} T_{ij}^k \xi_k$$

The numbers T_{ij}^k are called the structure constants of the Lie algebra in that basis.

Exercise 3.1. What are the dimensions of $\mathrm{sl}(n)$, $\mathrm{so}(n)$, $\mathrm{sp}(2n)$ and $\mathrm{so}(p, q)$?

Lecture 4

Lie Groups in Control – Examples

4.1. Example of Lie brackets in a bilinear control system

Consider the bilinear control system

$$\dot{x} = uAx + vBx, \tag{4.1}$$

where u and v are controls and A, B are constant $n \times n$ matrices. Consider the choice of controls depicted in the graphs in Figure 4.1.

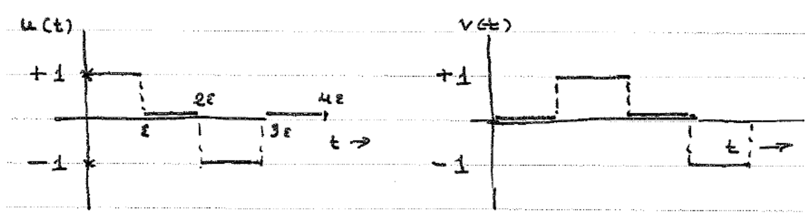


Figure 4.1: Control signals for the bilinear system.

The corresponding evolution in state space after 4ϵ time units is given by

$$\begin{aligned} x(4\epsilon) &= e^{-\epsilon B}\, e^{-\epsilon A}\, e^{\epsilon B}\, e^{\epsilon A}\, x_0 \\ &= \Big(I - \epsilon B + \frac{\epsilon^2}{2!}B^2 + \cdots\Big)\Big(I - \epsilon A + \frac{\epsilon^2}{2!}A^2 + \cdots\Big)\Big(I + \epsilon B + \frac{\epsilon^2}{2!}B^2 + \cdots\Big)\Big(I + \epsilon A + \frac{\epsilon^2}{2!}A^2 + \cdots\Big)x_0. \end{aligned}$$

Multiplying terms yields,

$$\begin{aligned} x(4\epsilon) &= \left(I - \epsilon(A+B) + \epsilon^2\left(\frac{B^2}{2!} + \frac{A^2}{2!} + BA\right) + \cdots\right)\left(I + \epsilon(A+B) + \epsilon^2\left(\frac{B^2}{2!} + \frac{A^2}{2!} + BA\right) + \ldots\right) x_0 \\ &= \left(I - \epsilon^2\left(A^2 + B^2 + AB + BA\right) + \epsilon^2\left(B^2 + A^2 + 2BA\right) + \cdots\right) x_0 \\ &= x_0 + \epsilon^2\left(BA - AB\right) x_0 + o(\epsilon^2). \end{aligned}$$

(Recall 'little $o(\cdot)$' notation: we say that $f(\sigma)$ is $o(\sigma)$ if $\lim_{\sigma \to 0} \frac{f(\sigma)}{\sigma} = 0$). Now we see that a gap may exist between the initial and ending locations, given by

$$x(4\epsilon) - x_0 = \epsilon^2\left(BA - AB\right) x_0 + o(\epsilon^2). \tag{4.2}$$

This gap is depicted in Figure 4.2.

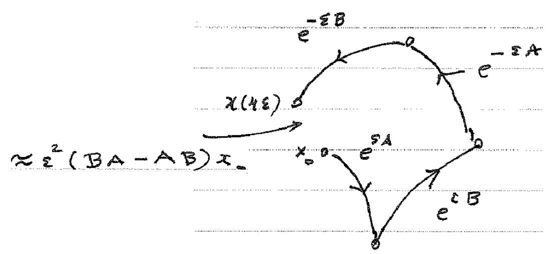


Figure 4.2: Illustration of difference in starting and ending locations due to switching.

Further, if $(BA - AB)$ is linearly independent of A and B, then we have a control (sequence) that generates an entirely new direction of motion, given by the Lie Bracket (matrix commutator)

$$[A, B] = BA - AB. \tag{4.3}$$

Additionally, this suggests that the controllability of bilinear control systems is related to Lie brackets.

4.2. Flows and Lie Derivatives

Nonlinear control systems modeled by differential equations also lead to the consideration of Lie brackets, as applied to vector fields. First, we consider the nonlinear system

$$\dot{x}(t) = f(x(t)). \tag{4.4}$$

This system is associated with a vector field $x \mapsto f(x)$ defined on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$). Suppose that for each $x_0 \in \mathbb{R}^n$, there is a unique solution $\phi(t, x_0)$ such that

$$\frac{d}{dt}\phi(t, x_0) = f\left(\phi(t, x_0)\right)$$

defined for all $t \in \mathbb{R}$ under suitable hypotheses—we will later prove the Cauchy-Lipschitz Existence-Uniqueness Theorem for $t \in (-\delta, \delta)$, $\delta > 0$.

It follows that the following properties hold for the unique solution $\phi(t, x_0)$:

(i) $\phi(0, x_0) = x_0$

(ii) $\phi(t, \phi(s, x_0)) = \phi(t + s, x_0), \qquad \forall t, s \in \mathbb{R}$, and $x_0 \in \mathbb{R}^n$

(iii) $\phi(-t, \phi(t, x_0)) = x_0$.

Thus $\{\phi(t, \cdot) : t \in \mathbb{R}^n\}$ defines a <u>one parameter group</u> of invertible maps with smooth inverses defined by the differential equation. It is customary to explicitly denote the dependence on f and refer to the <u>flow</u> of vector field f:

$$\{\Phi_t^f : \Phi_t^f(x_0) = \phi(t, x_0), \text{ satisfying } \frac{d}{dt}\phi(t, x_0) = f\left(\phi(t, x_0)\right)\}, \tag{4.5}$$

where $\Phi_t^f(x_0)$ denotes the solution at t of $\dot{x} = f(x)$ starting from $x_0 \in \mathbb{R}^n$.

The <u>effect</u> (action) of a vector field on a function ψ can be computed as follows:

Evaluate $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ on the trajectory generated by $\dot{x} = f(x)$, yielding $\psi(x(t)) = \psi \circ \Phi_t^f(x_0)$ for some initial condition x_0.

Then,

$$\begin{aligned} \frac{d}{dt}\psi(x(t)) &= \frac{\partial \psi}{\partial x}\Big|_{x(t)} \frac{dx(t)}{dt} \qquad \text{(chain rule)} \\ &= \frac{\partial \psi}{\partial x} f(x)\Big|_{x(t)}, \end{aligned}$$

where,

$$\frac{\partial \psi}{\partial x} = \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}, \cdots, \frac{\partial \psi}{\partial x_n}\right),$$

a row vector of partial derivatives of ψ with respect to the coordinates. We can also write,

$$\frac{d}{dt}\psi(x(t)) = \left(\sum_{i=1}^{n} f^i(x)\frac{\partial}{\partial x_i}\right)\psi\Big|_{x=x(t)}.$$

Letting,

$$L_f \triangleq \sum_{i=1}^{n} f^i(x)\frac{\partial}{\partial x_i} \tag{4.6}$$

denote the (first-order) Lie derivative operator, we can say a vector field f acts on a function ψ by Lie differentiation,

$$\psi \mapsto L_f\psi.$$

This view of how vector fields behave with respect to functions is key to understanding the Lie bracket of vector fields. Before considering the Lie bracket of two vector fields, first consider the quantity $(L_fL_g - L_gL_f)\psi$.
We have,

$$\begin{aligned}(L_fL_g - L_gL_f)\psi =& L_f(L_g\psi) - L_g(L_f\psi)\\ =& \sum_i f^i\frac{\partial}{\partial x_i}\left(\sum_j g^j\frac{\partial\psi}{\partial x_j}\right) - \sum_i g^i\frac{\partial}{\partial x_i}\left(\sum_j f^j\frac{\partial\psi}{\partial x_j}\right)\\ =& \sum_i f^i\sum_j\frac{\partial g^j}{\partial x_i}\frac{\partial\psi}{\partial x_j} + \sum_i f^i\sum_j g^j\frac{\partial^2\psi}{\partial x_i\partial x_j}\\ & - \sum_i g^i\sum_j\frac{\partial f^j}{\partial x_i}\frac{\partial\psi}{\partial x_j} - \sum_i g^i\sum_j f^j\frac{\partial^2\psi}{\partial x_i\partial x_j}\end{aligned}$$

Since mixed partial derivatives commute, the terms involving second derivatives cancel.
We have,

$$\begin{aligned}(L_fL_g - L_gL_f)\psi &= \left(\sum_j\left(\sum_i\frac{\partial g^j}{\partial x_i}f^i\right)\frac{\partial}{\partial x_j} - \sum_j\left(\sum_i\frac{\partial f^j}{\partial x_i}g^i\right)\frac{\partial}{\partial x_j}\right)\psi\\ &= \left(\sum_j\left(\left(\frac{\partial g}{\partial x}\right)f - \left(\frac{\partial f}{\partial x}\right)g\right)^j\frac{\partial}{\partial x_j}\right)\psi\end{aligned}$$

Thus we see that $L_fL_g - L_gL_f$ is simply the Lie derivative operator associated with the vector field

$$\left(\frac{\partial g}{\partial x}\right)f - \left(\frac{\partial f}{\partial x}\right)g$$

which we denote as $[f, g]$, the Lie bracket of two vector fields.

It follows that the operator commutator

$$[L_f, L_g] \triangleq L_f L_g - L_g L_f = L_{[f,g]}, \tag{4.7}$$

forms an operator $L_{[f,g]}$ by commutating the L_f and L_g operators.

The following properties hold for Lie Brackets involving vector fields f, g, h:

(i) $[f, g] = -[g, f]$

(ii) $[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$ where $\alpha, \beta \in \mathbb{R}$

(iii) (The Jacobi identity) $[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$

We may verify the Jacobi identity in a smart way by making use of the correspondence between vector fields and Lie derivative operators. Let ϕ be an arbitrary, differentiable test function.

$$\begin{aligned}
&[L_f, [L_g, L_h]]\phi + [L_h, [L_f, L_g]]\phi + [L_g, [L_h, L_f]]\phi \\
&\qquad = \quad (L_f L_g L_h - L_f L_h L_g - L_g L_h L_f + L_h L_g L_f)\phi \\
&\qquad\quad + (L_h L_f L_g - L_h L_g L_f - L_f L_g L_h + L_g L_f L_h)\phi \\
&\qquad\quad + (L_g L_h L_f - L_g L_f L_h - L_h L_f L_g + L_f L_h L_g)\phi \\
&\qquad \equiv \quad 0 \qquad \forall \quad \text{test functions}
\end{aligned}$$

This shows that the set of all Lie differentiation operators (L_f etc.) is a Lie algebra under operator commutation over the field of reals. Since the correspondence from vector fields to Lie derivative operators preserves brackets, and since the Lie derivative operators form a Lie algebra under operator commutation, it follows that the vector fields also form a Lie algebra under the bracket $[f, g]$ defined above.

The Lie bracket of vector fields is also referred to as the Jacobi-Lie bracket. Note that when $f(x) = Ax$ and $g(x) = Bx$ are linear vector fields, then

$$\begin{aligned}
[f, g] &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \\
&= BAx - ABx \\
&= (BA - AB)x \\
&= [B, A]x
\end{aligned}$$

where the $[\cdot, \cdot]$ is the last line denotes the Lie bracket in the Lie algebra $\mathrm{gl}(n, \mathbb{R})$.

The flow of vector field f satisfies

$$\left(\Phi_t^f\right)^{-1} = \Phi_{-t}^f = \Phi_t^{-f}$$

(reversing the arrow is the same as reversing the flow of time). For linear vector fields $f(x) = Ax$, and $g(x) = Bx$, using

$$\Phi_t^f(x) = e^{tA}x \quad \text{and} \quad \Phi_t^g(x) = e^{tB}x,$$

we have shown that

$$\begin{aligned}
\Phi_{-\epsilon}^g\left(\Phi_{-\epsilon}^f\left(\Phi_\epsilon^g\left(\Phi_\epsilon^f(x_0)\right)\right)\right) &= e^{-\epsilon B}\, e^{-\epsilon A}\, e^{\epsilon B}\, e^{\epsilon A}\, x_0 \\
&= x_0 + \epsilon^2\left(BA - AB\right)x_0 + o(\epsilon^2) \\
&= x_0 + \epsilon^2[f,g]x_0 + o(\epsilon^2)
\end{aligned}$$

In fact, the last line holds for general nonlinear vector fields. The proof of this relies on an expression for the flow from using the fundamental theorem of integral calculus.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable. Let $g(t) = F(x + th)$. Then,

$$\begin{aligned}
g(1) &= g(0) + \int_0^1 \frac{dg}{dt}dt \\
&= F(x) + \int_0^1 \frac{d}{dt}F(x+th)dt \\
&= F(x) + \int_0^1 DF(x+th)hdt \qquad \text{(Chain rule)},
\end{aligned}$$

where $DF(y)$ denotes the linear operator defined by

$$DF(y)\eta = \frac{d}{d\epsilon}F(y+\epsilon\eta)\bigg|_{\epsilon=0}. \tag{4.8}$$

(It is simply given by the Jacobian matrix, $DF(x) = \left[\frac{dF^i}{dx_j}\right]\Big|_x$). Recalling that $g(1) = F(x+h)$, we may now write,

$$F(x+h) = F(x) + \int_0^1 DF(x+th)hdt \tag{4.9}$$

Now the process can be repeated as follows:
Let $g(s) = DF(x+tsh)h$. Note the correspondence to the term within the integral in equation (4.9).

Then,

$$\begin{aligned}
g(1) &= DF(x+th)h \\
&= g(0) + \int_0^1 \frac{dg(s)}{ds} ds \\
&= DF(x)h + \int_0^1 \frac{d}{ds} DF(x+tsh)hds \\
&= DF(x)h + \int_0^1 D^2F(x+tsh)(h,h)tds \qquad \text{(chain rule)},
\end{aligned}$$

where $D^2F(x+tsh)(h,h)$ is a column vector with ith element given by

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F^i}{\partial x_j \partial x_k}\Big|_{x+tsh} h_j h_k.$$

Hence,

$$\begin{aligned}
F(x+h) &= F(x) + \int_0^1 \left(DF(x)\cdot h + \int_0^1 D^2F(x+tsh)(h,h)tds \right) dt \\
&= F(x) + tDF(x)h + \int_0^1\int_0^1 tD^2F(x+tsh)(h,h)dsdt \\
&= F(x) + tDF(x)h + \frac{t^2}{2!}D^2F(x)(h,h) + o(t^2)
\end{aligned}$$

(The last step can be completed by an additional application of the fundamental theorem after letting $g(\sigma) = D^2F(x+ts\sigma h)(h,h)$ and repeating the process.)

Applying this process to the flow Φ_t^f and using

$$\frac{d}{dt}\Phi_t^f(x) = f(\Phi_t^f(x)),$$

or equivalently,

$$\Phi_t^f(x) = x + \int_0^t f(\Phi_\sigma^f(x))d\sigma,$$

we obtain:

$$\Phi_t^f(x) = x + tf(x) + \frac{t^2}{2!}Df(x)x + o(t^2) \tag{4.10}$$

Exercise 4.1. Use this formula to prove the Lie-Trotter composition formula:

$$\boxed{\Phi_{-\epsilon}^g\left(\Phi_{-\epsilon}^f\left(\Phi_\epsilon^g\left(\Phi_\epsilon^f(x_0)\right)\right)\right) = x_0 + \epsilon^2[f,g]x_0 + o(\epsilon^2)}$$

(Hint: Carry along terms up to ϵ^2, and refer all values $f(x)$, $g(x)$, $Df(x)$, $Dg(x)$, back to $x = x_0$. Again for this purpose, use the fundamental theorem of integral calculus).

Example 4.1. Unicycle and Lie Brackets. The model of a unicycle,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} u\cos\theta \\ u\sin\theta \\ \omega \end{bmatrix},$$

can be written as a drift-free control system,

$$\dot{z} = uf(z) + wg(z),$$

where

$$f(z) = \begin{bmatrix} \cos(z_3) \\ \sin(z_3) \\ 0 \end{bmatrix} \qquad \textit{and} \qquad g(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, f and g are linearly independent vectors (directions of motion) at each z. Moreover,

$$\begin{aligned} [f,g] &= \frac{\partial g}{\partial z} f - \frac{\partial f}{\partial z} g \\ &= 0 \cdot f - \begin{bmatrix} -\sin(z_3) \\ \cos(z_3) \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(z_3) \\ \cos(z_3) \\ 0 \end{bmatrix}. \end{aligned}$$

This direction is linearly independent of f and g <u>at each z</u> as well. Thus, we obtain three independent directions of motion at each z: (1) using u alone, (2) using w alone, or (3) alternating peddling and steering. (This indicates controllability.)

Example 4.2. Non-holonomic integrator (R. Brockett).

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned}$$

$$f = \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix} \qquad \text{and} \qquad g = \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} \qquad \text{and} \qquad [f,g] = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix},$$

and the situation is the same as in the case of the unicycle.

Example 4.3. Kinematic Car (E. Nelson)
Consider a car represented schematically below.

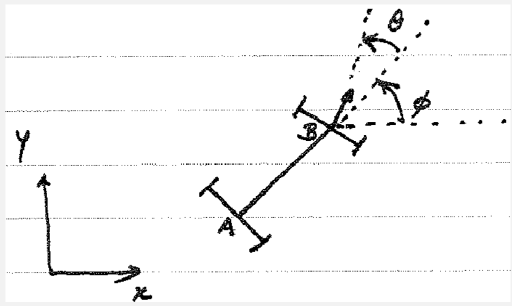


Figure 4.3: Kinematic model of a car.

$$
\begin{aligned}
B &\triangleq (x, y) \\
AB &= l \\
\theta &= \text{steering angle} \\
\phi &= \text{body orientation}
\end{aligned}
$$

Set $l = 1$ for convenience. There are two distinguished vector fields that a driver controls:

$$
\begin{aligned}
f &= \text{steer} = \frac{\partial}{\partial \theta} \qquad \text{(expressed via } L_f\text{)} \\
g &= \text{drive} = \cos(\phi + \theta)\frac{\partial}{\partial x} + \sin(\phi + \theta)\frac{\partial}{\partial y} + \sin(\theta)\frac{\partial}{\partial \phi} \\
&\quad \text{(expressed via } L_g\text{)}
\end{aligned}
$$

Define

$$
\begin{aligned}
\text{wriggle} &\triangleq [\text{steer}, \text{drive}] \\
&= -\sin(\phi + \theta)\frac{\partial}{\partial x} + \cos(\phi + \theta)\frac{\partial}{\partial y} + \cos(\theta)\frac{\partial}{\partial \phi} \\
\text{slide} &\triangleq -\sin(\phi)\frac{\partial}{\partial x} + \cos(\phi)\frac{\partial}{\partial y} \\
\text{rotate} &\triangleq \frac{\partial}{\partial \phi}
\end{aligned}
$$

Verify that at $\theta = 0$,

(i) $[\text{steer}, \text{drive}] = \text{slide} + \text{rotate}$

(ii) $[\text{steer}, \text{wriggle}] = -\text{drive}$

(iii) $[\text{wriggle}, \text{drive}] = \text{slide}$

Thus, steer, drive, $[\text{steer}, \text{drive}]$, and $[\text{drive}, [\text{steer}, \text{drive}]]$ give a set of linearly independent directions. Note also that slide has vanishing brackets with steer, drive, and wriggle.

Using the directions of motion outlined above, one can formulate a parallel-parking algorithm: wriggle, drive, $-$wriggle, $-$drive, repeat ...

Example 4.4. Pendulum with parametric amplification (R. Brockett)

This example is a model for a child pumping a swing.

See page 64 of R.W. Brockett, ”Nonlinear Systems and Differential Geometry,” Proc. IEEE, Vol 64, No 1, pp. 61-72, 1976.

This problem has a drift term. It also involves brackets of depth two, as in the case of the parking problem.

Lecture 5

Contraction Mapping, Existence & Uniqueness

In this lecture, we discuss the existence and uniqueness of solutions to ordinary differential equations. The central idea is the Contraction Mapping - Fixed-Point Theorem due to S. Banach.[1]

Definition 5.1. *Let (S, d) be a metric space and let $f : S \to S$ be a map. We say that f is a contraction if there exists $\rho \in (0, 1)$ such that*

$$d(f(x), f(y)) \leq \rho d(x, y) \qquad \forall x, y \in S.$$

△

Example 5.1. Let $S = \mathbb{R}^n$ and let $\|\cdot\|_\infty$ be defined by $\|x\|_\infty = \underset{1\leq i\leq n}{max}|x_i|$, $x \in \mathbb{R}^n$. Let $d(x, y) = \|x - y\|_\infty$. Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map (matrix) satisfying

$$|a_{ii}| > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}| \qquad \forall i \in \{1, 2, \ldots, n\}$$

(diagonal dominance). Then $\tilde{A} = D^{-1}(L + U)$ is a contraction, where D is a matrix containing only the diagonal elements of A, and L and U are matrices containing only the upper and lower components of A, respectively. We will re-visit this contraction in a later example.

[1]Stefan Banach was a central figure in the mathematical life of Poland in the pre-WWII era. See http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Banach.html.

Definition 5.2. *We say that $x^* \in S$ is a fixed point of a mapping $f : S \to S$ provided $f(x^*) = x^*$.* $\triangle$

The notion of a fixed point is important in economics (game theory) and many other fields.

Definition 5.3. *A sequence $\{x_k : k = 1, 2, \dots\} \subset S$ a metric space with metric d, is said to be convergent, if $\exists x^* \in S$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. In that case x^* is unique (proof: use the triangle inequality), and hence we can write $x^* = \lim_{n\to\infty} x_n$.* $\triangle$

Definition 5.4. *A sequence $\{x_k : k = 1, 2, \dots\} \subset S$ a metric space with metric d, is said to be a Cauchy sequence, if*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0.$$

$\triangle$

Exercise 5.1. Show that every convergent sequence is a Cauchy sequence. Note: the converse is NOT true in general.

Definition 5.5. *A metric space is said to be complete if every Cauchy sequence in S is convergent in S. (That is, if the converse of Exercise 5.1 holds.)* $\triangle$

Example 5.2. $S = \mathbb{R}$ with $d(x, y) = |x - y|$ is a complete metric space. Because of this example, $\mathbb{R}^n$ is also a complete metric space if we consider $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.

Norbert Wiener and (later) Stefan Banach focused attention on infinite dimensional vector spaces of functions that have a norm such that the associated metric is complete. Initially, these spaces came to be known as Wiener-Banach spaces, and now simply Banach spaces.

Given any norm $\|\cdot\|$ on a vector space V, we can associate a metric

$$d(x, y) = \|x - y\|, \qquad x, y \in V.$$

Theorem 5.1 (***Contraction Mapping - Fixed-Point Theorem***)

Let X be a Banach space and let $S \subset X$ be a closed subset. Let $f : S \to S$ be a mapping such that, for some $\rho \in (0, 1)$,

$$\|f(x) - f(y)\| \le \rho \|x - y\| \qquad \forall x, y \in S.$$

Then $\exists$ a unique fixed point $x^* \in S$ such that $f(x^*) = x^*$. Further, this fixed point can be obtained by that method of successive approximations (Banach iteration). $\square$

Before we proceed to the proof of Banach's theorem, we need a few basics.

Definition 5.6. *An open ball in a metric space (S, d) centered at $x_0 \in S$ and of radius $\epsilon > 0$ is denoted*

$$B_\epsilon(x_0) = \{x \in S : d(x, x_0) < \epsilon\}$$

$\triangle$

Definition 5.7. *We say a set $P \subset S$ is open (in a given metric) if given any $x \in P$, there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset P$.* $\triangle$

Definition 5.8. *A closed set has the property that for every convergent sequence $\{x_k : k = 1, 2, \dots\}$ contained in the set, the limit of the sequence x^* is also in the same set.* $\triangle$

Proof of Theorem 5.1

Proof of Banach's fixed-point theorem
Let $x_1 \in S$. Define the sequence $\{x_k : k \geq 1\}$ by $x_{k+1} = f(x_k)$. By hypothesis, $\{x_k\} \subset S$. Looking at the distance between successive elements,

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|f(x_k) - f(x_{k-1})\| \\
&\leq \rho \|x_k - x_{k-1}\| \\
&\leq \rho^2 \|x_{k-1} - x_{k-2}\| \qquad \text{(repeating the previous step)} \\
&\vdots \\
&\leq \rho^{k-1} \|x_2 - x_1\|
\end{aligned}$$

We can also look at the distance between two nonsuccessive terms, $\|x_{k+r} - x_k\|$, for $r \geq 1$,

$$\begin{aligned}
\|x_{k+r} - x_k\| &= \|x_{k+r} - x_{k+r-1} + x_{k+r-1} - x_{k+r-2} + x_{k+r-2} \cdots - x_k\| \\
&\leq \|x_{k+r} - x_{k+r-1}\| + \|x_{k+r-1} - x_{k+r-2}\| + \cdots + \|x_{k+1} - x_k\| \\
&\leq (\rho^{k+r-2} + \rho^{k+r-3} + \cdots + \rho^{k-1}) \|x_2 - x_1\| \qquad \text{for } k \geq 1 \\
&\leq \rho^{k-1} \sum_{j=0}^{\infty} \rho^j \|x_2 - x_1\| \\
&= \frac{\rho^{k-1}}{1 - \rho} \|x_2 - x_1\| .
\end{aligned}$$

Since $\rho < 1$, $\|x_{k+r} - x_k\| \to 0$ as $k \to \infty$. Hence, $\{x_k\}$ is a Cauchy sequence. Since X is a Banach space, there is an x^* such that $x_k \to x^*$. But S is closed.

Therefore, $x^* \in S$. To see that x^* is a fixed point,

$$\begin{aligned}\|x^* - f(x^*)\| &\le \|x^* - x_k\| + \|x_k - f(x^*)\| \\ &\le \|x^* - x_k\| + \rho\,\|x_{k-1} - x^*\| \\ &\to 0 \qquad \text{as} \quad k \to \infty\end{aligned}$$

Hence, $\|x^* - f(x^*)\| = 0 \implies x^* = f(x^*)$.
To prove uniqueness, suppose $y^* \in S$ is another fixed point.

$$\begin{aligned}\|x^* - y^*\| &= \|f(x^*) - f(y^*)\| \\ &\le \rho\,\|x^* - y^*\|\end{aligned}$$

But $\rho < 1$, so this only holds if $\|x^* - y^*\| = 0 \implies x^* = y^*$. ■

Remark 5.1. If the mapping were to depend on a parameter in a continuous way, so does the fixed point. △

Theorem 5.2 (*Continuity of a Fixed Point with respect to a Parameter*)

Let W be a metric space with metric d. Let X be a Banach space and let $S \subset X$ be a closed subset, such that $f : W \times S \to S$ has the following properties:

(i) Each partial map

$$f_\theta : S \to S \qquad \theta \in W$$

(defined by $f_\theta(x) = f(\theta, x)$) is a contraction with $\rho < 1$, independent of θ.

(ii) For each $x \in S$, the partial map

$$f^x : W \to S \qquad x \in S,$$

(defined by $f^x(\theta) = f(\theta, x)$), is continuous, (i.e. given $\epsilon > 0$ there exists $\delta^x > 0$ such that $d(\theta, \theta') < \delta^x \implies \|f^x(\theta - f^x(\theta')\| < \epsilon$.)

Then, the map $\theta \mapsto x^*_\theta$ which assigns to each $\theta \in W$, the (unique) fixed point x^*_θ of f_θ, is continuous. □

Proof of Theorem 5.2

Let x^*_θ and $x^*_{\theta'}$ be fixed points under $f_\theta(\cdot)$ and $f_{\theta'}(\cdot)$, respectively.

$$\begin{aligned}\|x^*_\theta - x^*_{\theta'}\| &= \|f_\theta(x^*_\theta) - f_{\theta'}(x^*_{\theta'})\| \\ &\le \|f_\theta(x^*_\theta) - f_\theta(x^*_{\theta'})\| + \|f_\theta(x^*_{\theta'}) - f_{\theta'}(x^*_{\theta'})\| \\ &\le \rho\,\|x^*_\theta - x^*_{\theta'}\| + \left\|f^{x^*_{\theta'}}(\theta) - f^{x^*_{\theta'}}(\theta')\right\| \qquad \text{(since } f_\theta(x) = f(\theta, x) = f^x(\theta)\text{)}.\end{aligned}$$

Hence, we have

$$\|x^*_\theta - x^*_{\theta'}\| \le \frac{1}{1-\rho}\left\|f^{x^*_{\theta'}}(\theta) - f^{x^*_{\theta'}}(\theta')\right\|.$$

However, since $f^x(\theta)$ is continuous in θ, we have

$$d(\theta,\theta') < \delta^{x^*_\theta} \implies \left\| f^{x^*_{\theta'}}(\theta) - f^{x^*_{\theta'}}(\theta') \right\| < \epsilon.$$

Therefore,

$$d(\theta,\theta') < \delta^{x^*_\theta} \implies \|x^*_\theta - x^*_{\theta'}\| < \frac{\epsilon}{1-\rho}.$$

This proves the continuity of the fixed point with respect to the parameter θ. ■

Example 5.3 (Jacobi's Algorithm)**.** The linear equation in $\mathbb{R}^n$,

$$Ax = b$$

where A is a square matrix can be identified as the fixed-point problem

$$x = -D^{-1}(L+U)x + D^{-1}b$$

where $A = L + D + U$ denotes the decomposition into strictly lower triangular, diagonal, and strictly upper triangular parts. We assume D is invertible. Jacobi's algorithm to solve this problem,

$$x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b,$$

is a special case of Banach iteration. To guarantee convergence, it is sufficient that A be diagonally dominant:

$$|a_{ii}| > \sum_{\substack{j \neq i \\ j=1}}^{n} |a_{ij}| \qquad \forall i \in \{1, 2, \ldots, n\}.$$

Then we can take $\rho = \max_i \left(\frac{1}{|a_{ii}|} \sum_{j\neq i}^{n} |a_{ij}| \right)$, making $f(x) = -D^{-1}(L+U)x + D^{-1}$, a contraction on all of $\mathbb{R}^n$.

Example 5.4. Consider the scalar equation

$$g(x) = x^2 - b = 0 \qquad b > 0$$

Let $y = 1 - x$. The problem of finding the (positive) square root of b is a fixed-point problem,

$$y = \frac{1}{2}[(1-b) + y^2] = f(y).$$

Suppose $|1-b| < \rho < 1$. Then f maps the closed subset $S = \{y : |y| \leq \rho\} \subset \mathbb{R}$ into itself and it is a contraction on S with parameter ρ. Thus, the algorithm

$$y_{k+1} = \frac{1}{2}[(1-b) + y_k^2]$$

converges for $|1-b| \leq \rho < 1$. It is equivalent to

$$x_{k+1} = x_k - \frac{1}{2}x_k^2 + \frac{1}{2}b.$$

Exercise 5.2. How does this compare with Newton's algorithm?

We are interested in (and ready for) applying Banach's theorem to ordinary differential equations.

Let

$$\dot{x} = f(t, x) \tag{5.1}$$

be a non-autonomous ordinary differential equation. A continuously differentiable solution $x(t)$ is,

$$x(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma)) d\sigma \tag{5.2}$$

for $t \in [t_0, t_0 + \delta]$ for some $\delta > 0$. We aim to show existence and uniqueness to the above integral equation in a suitable function space, the space $(X, \|\cdot\|_X)$ below.

For any $\delta > 0$, the space

$$X = \{\Psi : [t_0, t_0 + \delta] \to \mathbb{R}^n | \Psi \text{ continuous}\}$$

with norm

$$\|\Psi\|_X = \max_{t \in [t_0, t_0+\delta]} \|\Psi(t)\| \tag{5.3}$$

where $\|\cdot\|$ in $\mathbb{R}^n$ is any norm, is a complete normed linear space (i.e. Banach space).

Exercise 5.3. Prove the completeness of space X. $\to$ See Appendix B of Khalil 3rd ed.

Theorem 5.3 (*Local Existence and Uniqueness*)

Consider the system in (5.1). *Let f be piecewise continuous in t and satisfy the Lipschitz condition,*

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\| \tag{5.4}$$

$\forall x, y \in B_r(x_0) = \{x : \|x - x_0\| \leq r\}$ and $\forall t \in [t_0, t_1]$.

Then, there is some $\delta > 0$ such that the integral equation (5.2) with $t \in [t_0, t_0 + \delta]$ has a unique solution x in X. It is differentiable with respect to t and $\dot{x}(t)$ agrees with $f(t, x(t))$ at all points of continuity of f. □

Proof of Theorem 5.3

Define $P : X \to X$

$$(Px)(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))d\sigma \qquad t \in [t_0, t_0 + \delta].$$

Let $x_0(\cdot)$ denote the constant function belonging to X, $x_0(t) \equiv x_0$ for $t \in [t_0, t_0 + \delta]$.

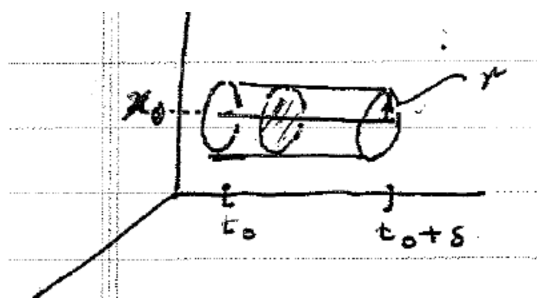


Figure 5.1: Solid tube

Let $S \triangleq \{x \in X : \|x - x_0\|_X \leq r\}$, the solid tube shown in Figure 5.1. It is a closed ball in X. We will be choosing $\delta > 0$ such that $t_0 + \delta \leq t_1$.

We now make a series of Observations:

(i) Since f is piecewise continuous in t, so is $\|f(t, x)\|$ for every x. Thus $\|f(t, x_0)\|$ is bounded on $[t_0, t_1]$. We set

$$h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\| .$$

(ii) P *maps* S *into* S*. To see this, let* $x(\cdot) \in S$*. Then for* $t \leq t_0 + \delta$*,*

$$\begin{aligned}
\|(Px)(t) - x_0\| &= \left\| \int_{t_0}^{t} f(\sigma, x(\sigma)) d\sigma \right\| \\
&\leq \int_{t_0}^{t} \|f(\sigma, x(\sigma))\| \, d\sigma \\
&\leq \int_{t_0}^{t} \|f(\sigma, x(\sigma)) - f(\sigma, x_0)\| \, d\sigma + \int_{t_0}^{t} \|f(\sigma, x_0)\| \, d\sigma \\
&\qquad\qquad \textit{(triangle inequality)} \\
&\leq \int_{t_0}^{t} (L \|x(\sigma) - x_0\| + h) d\sigma \\
&\qquad\qquad \textit{(Lipschitz condition and observation (i))} \\
&\leq \int_{t_0}^{t} (Lr + h) d\sigma \qquad \textit{(since } x(\cdot) \in S\textit{)} \\
&= (t - t_0)(Lr + h) \\
&\leq \delta(Lr + h).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Px - x_0\| &= \max_{t \in [t_0, t_0 + \delta]} \|(Px)(t) - x_0\| \\
&\leq \delta(Lr + h) \\
&\leq r \qquad \textit{if} \qquad \delta \leq \frac{r}{Lr + h}
\end{aligned}$$

So choosing $\delta \leq \frac{r}{Lr+h}$ *ensures that* P *maps* S *into* S*.*

(iii) P *is a contraction on* S*. To see this, let* $x, y \in S$*.*

$$\begin{aligned}
\|(Px)(t) - (Py)(t)\| &= \left\| \int_{t_0}^{t} [f(\sigma, x(\sigma)) - f(\sigma, y(\sigma))] d\sigma \right\| \\
&\leq \int_{t_0}^{t} \|f(\sigma, x(\sigma)) - f(\sigma, y(\sigma))\| \, d\sigma \\
&\leq \int_{t_0}^{t} L \|x(\sigma) - y(\sigma)\| \, d\sigma \qquad \textit{(Lipschitz condition)} \\
&\leq L(t - t_0) \|x(\cdot) - y(\cdot)\|_X
\end{aligned}$$

Hence,

$$\begin{aligned}
\|Px - Py\|_X &\leq L\delta \|x - y\|_X \\
&\leq \rho \|x - y\|_X \qquad \textit{if} \qquad \delta \leq \frac{\rho}{L}
\end{aligned}$$

Thus, choosing $\rho < 1$, and

$$\delta \leq \min\left(t_1 - t_0, \frac{r}{Lr+h}, \frac{\rho}{L}\right)$$

ensures that $P : S \to S$ is a contraction mapping.

Hence, by the Contraction-Mapping Fixed-Point Theorem of Banach, there is a unique fixed point $P \in S$, the solution to the integral equation. We can actually show that this is the only solution in X.

Since $x_0 \in B(x_0, r)$, any (continuous solution $x(t)$ must lie inside $B(x_0, r)$ for some nontrivial interval of time. Suppose $x(t)$ leaves $B(x_0, r)$ and $t_0 + \mu$ is the first instant of time that $x(t)$ intersects $\partial B(x_0, r)$ the boundary of $B(x_0, r)$. Then

$$\|x(t_0 + \mu) - x_0\| = r.$$

On the other hand, $\forall t \leq t_0 + \mu$,

$$\|x(t) - x_0\| \leq \int_{t_0}^{t} (Lr + h) ds \qquad (\text{see observation (ii)}),$$

so that

$$\begin{aligned} r &= \|x(t_0 + \mu) - x_0\| \\ &\leq (Lr + h)\mu \qquad \Longrightarrow \qquad \mu \geq \frac{r}{Lr+h} \geq \delta. \end{aligned}$$

Hence the solution starting at x_0 satys in $B(x_0, r)$ and hence in S during $[t, t_0 + \delta]$. Consequently, uniqueness of the solution in S implies uniqueness of the solution in X. ■

Remark 5.2. Here, Banach iteration = Picard-Lindelof iteration. △

Notice that the map P in the local existence and uniqueness theorem depends on x_0 in a continuous way.

Corollary 5.1

Let W be a metric space that parametrizes a family of differential equations and initial conditions. Suppose the parametrization is such that the conditions of the Theorem 5.2 are satisfied. Then by Theorem 5.2, the solution obtained in the local existence and uniqueness theorem, depends continuously on x_0 and more generally on $\theta \in W$. □

This corollary is a very useful result to keep in mind. The following lemma leads to comparison of solutions, and informally provides a method to “solve” an inequality.

Lemma 5.1 (*Gronwall-Bellman Inequality*)

Let $\lambda : [a,b] \to \mathbb{R}$ be continuous and $\mu : [a,b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a,b] \to \mathbb{R}$ satisfies the implicit inequality

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds, \qquad a \leq t \leq b,$$

then it also satisfies the explicit inequality

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s)\exp\left[\int_s^t \mu(\tau)d\tau\right] ds, \qquad a \leq t \leq b. \tag{5.5}$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda \exp\left[\int_\sigma^t \mu(\tau)d\tau\right].$$

If in addition, $\mu(t) \equiv \mu \geq 0$ is a constant, then

$$y(t) \leq \lambda \exp\left[\mu(t-a)\right].$$

□

Proof of Lemma 5.1

Let

$$z(t) = \int_a^t \mu(s)y(s)ds$$
$$v(t) = z(t) + \lambda(t) - y(t) \geq 0.$$

Then, z is differentiable and

$$\begin{aligned} \dot{z}(t) &= \mu(t)y(t) \\ &= \mu(t)z(t) + \mu(t)\lambda(t) - \mu(t)v(t). \end{aligned}$$

This scalar equation has the solution

$$z(t) = \int_a^t \phi(t,s)[\mu(s)\lambda(s) - \mu(s)v(s)]ds \qquad (\text{since } z(a) = 0),$$

where

$$\phi(t,s) = \exp\left[\int_s^t \mu(\tau)d\tau\right] > 0$$

By hypothesis, $\int_a^t \phi(t,s)\mu(s)v(s)ds \geq 0$. Therefore,

$$z(t) \leq \int_a^t \exp\left[\int_s^t \mu(\tau)d\tau\right] \cdot \lambda(s)\mu(s)ds$$

and since $y(t) \leq \lambda(t) + z(t)$, the proof of the general case is completed.

The remaining cases amount to computing integrals—left to the reader. ■

Corollary 5.2

$f(t,x)$ is piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with Lipschitz constant L, where $W \subset \mathbb{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y) \quad \text{with} \quad y(t_0) = y_0,$$

and

$$\dot{z} = f(t,x) + g(t,z) \quad \text{with} \quad z(t_0) = z_0,$$

such that $y(t), z(t) \in W, \quad \forall t \in [t_0, t_1]$.

Suppose the perturbation is bounded:

$$\|g(t,x)\| \leq \mu \qquad \forall (t,x) \in [t_0, t_1] \times W$$

for some $\mu \geq 0$, and $\|y_0 - z_0\| \leq \gamma$.

Then,

$$\|y(t) - z(t)\| \leq \gamma \exp\left[L(t - t_0)\right] + \frac{\mu}{L}\left(\exp\left[L(t-t_0)\right] - 1\right) \qquad \forall t \in [t_0, t_1].$$

□

Proof of Corollary 5.1

$$\begin{aligned} y(t) &= y_0 + \int_{t_0}^{t} f(s, y(s))ds \\ z(t) &= z_0 + \int_{t_0}^{t} [f(s, z(s)) + g(s, z(s))]ds. \end{aligned}$$

Then,

$$\begin{aligned} \|y(t) - z(t)\| &\leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(s, y(s)) - f(s, z(s))\| \, ds + \int_{t_0}^{t} \|g(s, z(s))\| \, ds \\ &\leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L \, \|y(s) - z(s)\| \, ds. \end{aligned}$$

By the Gronwall-Bellman inequality,

$$\begin{aligned}\|y(t)-z(t)\| &\le \gamma+\mu(t-t_0)+\int_{t_0}^{t} L\cdot(\gamma+\mu(s-t_0))\exp[L(t-t_0]ds\\ &= \gamma+\mu(t-t_0)-\gamma-\mu(t-t_0)+\gamma\exp[L(t-t_0)]\\ &\quad +\int_{t_0}^{t}\mu\cdot\exp[L(t-s)]ds \qquad (\textit{integration by parts}).\\ &= \gamma\exp[L(t-t_0)]+\frac{\mu}{L}\left(\exp[L(t-t_0)]-1\right)\end{aligned}$$

■

Remark 5.3. In application as in the original Gronwall-Bellman inequality, one is turning an implicit inequality explicit—in effect "solving the inequality." △

Remark 5.4. Corollary 5.2 allows us to quantitatively estimate the effects of perturbations—in initial conditions and in the dynamics. Such estimates are useful to keep in mind—all models of physical systems display errors due to various unavoidable approximations. △

Theorem 5.4 (*Global Existence and Uniqueness*)

Suppose $f(t,x)$ in the local existence and uniqueness theorem is

(i) piecewise continuous in t,

(ii) satisfies

$$\|f(t,x_0)\| \le h, \quad \text{and}$$

(iii) satisfies the global Lipschitz condition

$$\|f(t,x)-f(t,y)\| \le L\cdot\|x-y\| \qquad \forall x,y\in\mathbb{R}^n, t\in[t_0,t_1],$$

then

$$\dot{x}(t)=f(t,x) \quad \text{with} \quad x(t_0)=x_0,$$

has a unique solution on $[t_0,t_1]$. □

Proof of Theorem 5.4

We show how to modify the proof local existence and uniqueness as required. We now let r be arbitrarily large (due to the global Lipschitz condition) so that

$$\frac{r}{Lr+h} > \frac{\rho}{L} \qquad (\text{by taking } r > \frac{\rho h}{(1-\rho)L}).$$

Thus, we only need $\delta \le \min\{t_1-t_0, \frac{\rho}{L}\}$ for $\rho<1$. If $t_1-t_0 \le \frac{\rho}{L}$, we would let $\delta = t_1-t_0$ and we are done.

If not, choose $\delta = \frac{\rho}{L}$, divide $[t_0,t_1]$ into a finite number of subintervals of length $\delta = \frac{\rho}{L}$ and repeat that many more times, applying the arguments of the *local existence and uniqueness theorem. This completes the proof.* ■

Example 5.5.

$$\dot{x} = -x^3$$

does not satisfy the global Lipschitz condition, but there is a unique solution,

$$x(t) = \text{sgn}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}} \qquad \forall t \geq t_0.$$

The essential idea here is that if $x(0) = a$, the set $\{x : |x| \leq a\}$ is a positively invariant, closed and bounded set for the dynamics $\dot{x} = -x^3$. This idea can be generalized in the following theorem.

Theorem 5.5 (*Long-Time Existence and Uniqueness*)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz on a domain $D \subset \mathbb{R}^n$. Suppose there is a closed and bounded set $W \subset D$ such that $x(0) = x_0 \in W$ and $f\big|_{\partial W}$ points into W. Then there is a unique solution $x(t)$ to $\dot{x} = f(x)$ such that $x(0) = x_0$. □

Proof of Theorem 5.5

Left as an exercise. ■

Exercise 5.4. Prove the Long-Time Existence-Uniqueness Theorem.

5.1. Definitions and Properties of the Lipschitz Condition

Definition 5.9. *f is locally Lipschitz on a domain (an open and connected set) $D \subset \mathbb{R}^n$ if each point p of D has a neighborhood (i.e. ball $B_\epsilon(p)$ surrounding p, $\epsilon > 0$) such that*

$$\|f(x) - f(y)\| \leq L_{B_\epsilon} \cdot \|x - y\| \qquad \forall x, y \in B_\epsilon \quad \textit{for some} \quad L_{B_\epsilon} > 0. \tag{5.6}$$

△

Definition 5.10. *f is Lipschitz on a set W if*

$$\|f(x) - f(y)\| \leq L \cdot \|x - y\| \qquad \forall x, y \in W \quad \textit{for some} \quad L > 0.$$

△

Properties

(a) f is locally Lipschitz on a domain D implies that f is continuous on D.

(b) f is Lipschitz on domain D implies that f is uniformly continuous on D.

(c) The converse of (a) is NOT true.

(d) f is locally Lipschitz on domain D does NOT imply that f is Lipschitz on D (due to the lack of uniformity of the Lipschitz constant).

(e) f is locally Lipschitz on domain D implies that f is Lipschitz on every closed and bounded subset of D.

(f) f is continuously differentiable implies that f is locally Lipschitz. The converse is em far from true.

Some of these properties can be summarized and easily remembered by recognizing that continuous differentiability is stronger than local Lipschitz, and local Lipschitz is in turn stronger than continuity.

Lecture 6

Mean Value Theorem

One of the basic results of single variable calculus is the classical Mean Value Theorem (MVT). In this lecture, we derive the MVT for higher dimensions, highlight its importance, and use it to derive a lemma (previously stated) that relates continuous differentiability and the local Lipschitz condition.

Theorem 6.1 (*MVT*)

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on interval (a, b). There exists c, $a < c < b$ such that the derivative

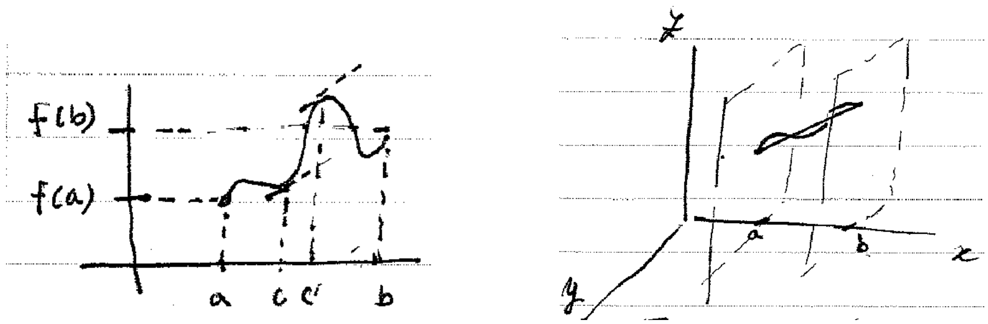
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Proof of Theorem 6.1

See an elementary text on single variable calculus. ■

Figure 6.1(a) gives us a picture of what is going on. The essential geometric idea is that at c (and c'), the tangent to the graph of f is parallel to the line joining point $(a, f(a))$ and $(b, f(b))$.



(a) MVT

(b) “Cork screw” curve. (No MVT).

Figure 6.1: Mean Value Theorem illustrations.

Let us see what happens in higher dimensions. Consider

$$\begin{aligned} f : [a,b] &\to \mathbb{R}^2 \\ x &\mapsto f(x) = (y,z) \end{aligned}$$

If the curve defined by f is of the “cork screw” variety (see Figure 6.1(b)), then there is no c, $a < c < b$ at which the tangent to the curve is parallel to the line adjoining points $(a, f(a))$ and $(b, f(b))$ in the x–y–z space. The classical MVT does not hold. The following is a specific example.

Example 6.1.

$$\begin{aligned} f : \left[0, \frac{\pi}{2}\right] &\to \mathbb{R}^2 \\ x &\mapsto (\cos(x), \sin(x)) \end{aligned}$$

$$\begin{aligned} f(b) - f(a) &= (-1, 1) \in \mathbb{R}^2 \\ b - a &= \frac{\pi}{2}. \end{aligned}$$

There does not exist a $c \in [0, \frac{\pi}{2}]$ such that $\frac{\pi}{2}(-\sin(c), \cos(c)) = (-1, 1)$ since it would require

$$\sin^2(c) + \cos^2(c) = \frac{8}{\pi} \neq 1.$$

The correct form of the mean value theorem in higher dimensions is actually an inequality. We need some preliminary results first though.

Lemma 6.1

Let $f : [a,b] \to V$ for $a < b$ where V is a normed linear space and $g : [a,b] \to \mathbb{R}$, with f and g continuous on $[a,b]$ and differentiable on (a,b). Sup-

pose, $\|f'(t)\| \le g'(t)$ for $a < t < b$. Then,

$$\|f(b) - f(a)\| \le g(b) - g(a).$$

□

Proof of Lemma 6.1

$$\begin{aligned}\|f(b) - f(a)\| &= \left\|\int_a^b f'(\sigma)d\sigma\right\| \\ &\le \int_a^b \|f'(\sigma)\|\, d\sigma \\ &\le \int_a^b g'(\sigma)d\sigma \\ &= g(b) - g(a)\end{aligned}$$

■

Lemma 6.2

Same hypotheses as in Lemma 6.1, except that the condition on existence and inequality of derivatives holds for all $t \in [a, b]$, except for a countable set of points. Same conclusion as Lemma 6.1. □

Proof of Lemma 6.2

Essentially the same argument as Lemma 6.1 since the integrals are unaffected.

■

Corollary 6.1

Same hypotheses on f as in Lemma 6.1, and $g(t) = kt$, $k > 0$.
(Thus, $\|f'(x)\| \le k \quad \forall t \in (a, b)$).
Then,

$$\|f(b) - f(a)\| \le k(b - a).$$

□

Proof of Corollary 6.1

Substitute $g(a) = ka$ and $g(b) = kb$. ■

We need the definition of derivative for maps.

Definition 6.1. *Let E, F be normed linear spaces over $\mathbb{R}$. Let $U \overset{open}{\subset} E$. Suppose $f : U \to F$. We say that f is differentiable at $a \in U$ if there is a continuous linear map $L : E \to F$ such that*

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - Lh\|_F}{\|h\|_E} = 0.$$

Here h is such that $h + a \in U$. Clearly, if L exists it is unique and given by,

$$\begin{aligned} L(k) &= \lim_{t \to 0} \frac{f(a+kt) - f(a)}{t} \\ &= \frac{d}{dt} f(a+kt)\Big|_{t=0} \end{aligned}$$

We call L the derivative of f at a and denote it by $(Df)_a$ and sometimes by $Df(a)$. $\triangle$

Exercise 6.1 (Chain rule)**.** Prove

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a),$$

where the left-hand side composition denotes the composition of two nonlinear maps and the right-hand side composition denotes the composition of two linear maps.

Exercise 6.2 (Jacobian)**.** Prove that if $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$,

$$(Df)(a) \cdot h = \left[\frac{\partial f^i}{\partial x_j}\right] h,$$

where $\left[\frac{\partial f^i}{\partial x_j}\right]$ is the Jacobian matrix.

Theorem 6.2 (*Mean Value Theorem for Maps*)

Let $f : U \overset{\text{open}}{\subset} E \to F$ be a map of normed linear spaces. Let $[a, b]$ denote the line segment $\{(1-t)a + tb : 0 \le t \le 1\}$, with end-points a, b contained in U. Then,

$$\|f(b) - f(a)\| \le \sup_{0 \le t \le 1} \|Df[(1-t)a + tb]\| \cdot \|b - a\|. \tag{6.1}$$

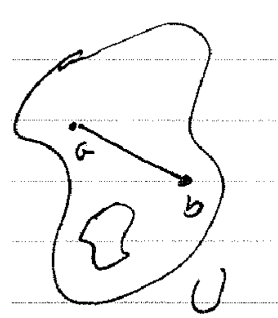


Figure 6.2: MVT for maps illustration.

□

Proof of Theorem 6.2

Simply restrict f to the line segment $[a, b]$ and then apply Corollary 6.1 above.

■

Another useful result from calculus is the Fundamental Theorem of Integral Calculus

Theorem 6.3 (*Fundamental Theorem of Integral Calculus*)

Let X, Y be Banach spaces. Let $U \overset{open}{\subset} X$ and $f : U \to Y$ be a differentiable map everywhere in U, or a C^1 map. Suppose $x + ty \in U \; \forall t \in [0, 1]$. Then

$$f(x + y) = f(x) + \int_0^1 Df(x + ty)y dt. \tag{6.2}$$

□

Proof of Theorem 6.3

The completeness / Banach property is used in the proper definition of the integral with all attendant properties, as in single variable calculus. We take this for granted. Now, set

$$g(t) = f(x + ty) \qquad 0 \le t \le 1.$$

By the chain rule, for $0 < t < 1$,

$$g'(t) = Df(x + ty)y$$

Let

$$h(t) = f(x) + \int_0^t Df(x + sy)yds \qquad 0 \le t \le 1.$$

Then,

$$h'(t) = Df(x + ty)y \qquad 0 < t < 1.$$

Hence, $g'(t) = h'(t)$, which implies that $g(t) = h(t) +$ constant, for $0 < t < 1$.

By continuity of g and h (they are integrals), $g(t) = h(t) +$ constant, for $0 \le t \le 1$.

But

$$g(0) = h(0) = f(x),$$

and

$$g(1) = h(1) = f(x + y).$$

So

$$f(x + y) = f(x) + \int_0^1 Df(x + ty)y dt.$$

■

Lemma 6.3 (*Continuous Differentiability implies Locally Lipschitz*)

Let $f : [a, b] \times D \to \mathbb{R}^n$, for domain $D \subset \mathbb{R}^n$, continuous in t and $\left(\frac{\partial f}{\partial x}\right)$ exists and is continuous on $[a, b] \times D$. Then f is locally Lipschitz on $[a, b] \times D$. □

Proof of Lemma 6.3

For $x_0 \in D$, let $r > 0$ be such that

$$B_r = \{x : \|x - x_0\| \leq r\} \subset D.$$

B_r is closed and bounded. B_r is convex, since for $x_1, x_2 \in B_r$ and $0 \leq \alpha \leq 1$,

$$\begin{aligned} \|\alpha x_1 + (1-\alpha)x_2 - x_0\| &= \|\alpha x_1 + (1-\alpha)x_2 - \alpha x_0 - (1-\alpha)x_0)\| \\ &= \|\alpha(x_1 - x_0) + (1-\alpha)(x_2 - x_0)\| \\ &\leq \alpha \|x_1 - x_0\| + (1-\alpha) \|x_2 - x_0\| \\ &\leq \alpha r + (1-\alpha) r \\ &= r. \end{aligned}$$

Then, by the MVT for maps, $\forall x, y \in B_r$

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leq \sup_{0 \leq s \leq 1} \|Df[(1-s)x + sy]\| \cdot \|x - y\| \\ &\leq \sup_{t \in [a,b]} \sup_{0 \leq s \leq 1} \|Df[(1-s)x + sy]\| \cdot \|x - y\| \\ &= L \|x - y\|, \end{aligned}$$

where we used continuity with respect to both t and x of Df in the $\sup$ norms. ■

Even though we have shown that the proper MVT in higher dimensions is an inequality, the following theorem is special case in which the MVT has a familiar form.

Theorem 6.4

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 at each point x of an open set $S \subset \mathbb{R}^n$. Suppose $x^*, y^* \in S$ are such that the line segment $L(x^*, y^*)$ joining x^*, y^* lies entirely in S. Then there exists a point $x \in L(x^*, y^*)$ such that

$$f(y^*) - f(x^*) = \left(\frac{\partial f}{\partial x}\right)_{x=z} (y^* - x^*) \tag{6.3}$$

□

Proof of Theorem 6.4

Let

$$\begin{aligned} g(t) &= f((1-t)x^* + ty^*) \\ g(0) &= f(x^*) \\ g(1) &= f(y^*) \\ g'(t) &= \left(\frac{\partial f}{\partial x}\right)\Big| \qquad \frac{d}{dt}((1-t)x^* + ty^*) \end{aligned}$$

By the scalar MVT (Theorem 6.1),

$$g(1) - g(0) = g'(t)\Big|_{t=t^*}(1-0)$$

which is the same as saying

$$f(y^*) - f(x^*) = \left(\frac{\partial f}{\partial x}\right)_{x=z}(y^* - x^*)$$

where $z = (1-t^*)x^* + t^*y^* \in L(x^*, y^*)$. ■

Lecture 7

Planar Systems

7.1. Linear Planar Setting

The study of planar systems is best explored by the study of planar linear systems. Consider the system in $\mathbb{R}^2$,

$$\dot{x} = Ax \tag{7.1}$$

where A is a 2×2 real constant matrix. From linear algebra, there is a real, nonsingular matrix P such that

$$J = PAP^{-1} \tag{7.2}$$

is one of the following three forms:

(i)

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{7.3}$$

(ii)

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \tag{7.4}$$

(iii)

$$J = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \tag{7.5}$$

These are the possible real Jordan forms.

The change of variables $y = Px$ defines the system

$$\dot{y} = Py \tag{7.6}$$

with solutions related by the formulas

$$
\begin{aligned}
x(t) &= P^{-1}y(t) \\
&= P^{-1}\exp(Jt)y(0) \\
&= P^{-1}\exp(Jt)Px(0).
\end{aligned}
$$

Everything about the behavior of x can be determined from that of $y(\cdot)$. In case (i),

$$
y_i(t) = \exp(\lambda_i t)y_i(0) \qquad i = 1, 2.
$$

In case (ii),

$$
\begin{aligned}
y_1(t) &= \exp(\lambda t)(y_1(0) + ty_2(0)) \\
y_2(t) &= \exp(\lambda t)y_2(0)
\end{aligned}
$$

In case (iii), letting $r = \sqrt{y_1^2 + y_2^2}$ and $\phi = \arctan(y_2/y_1)$, we get

$$
\begin{aligned}
\dot{r} &= \alpha r \\
\dot{\phi} &= -\beta
\end{aligned}
$$

Thus, in this last case, r spirals in or out of 0 accordingly if $\alpha < 0$ or $\alpha > 0$.

The behavior of the linear system around the origin is thus captured by the classification in Figure 7.1, up to a nonsingular transformation P. In the figure, we represent the behavior of (y_1, y_2).

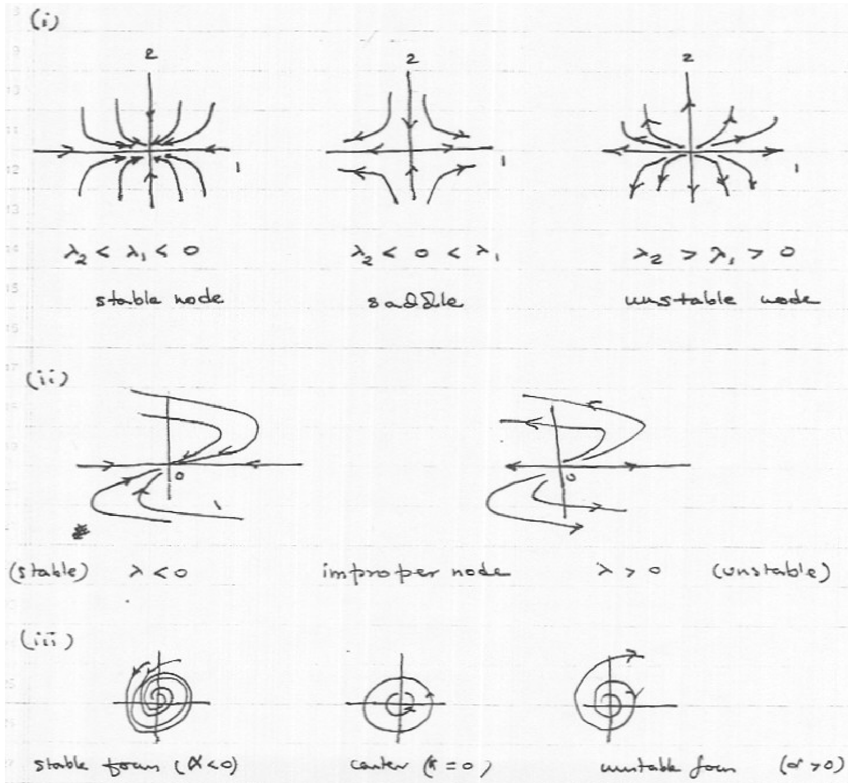


Figure 7.1: Classifications of Planar/Linear Systems.

It is clear that a small perturbation of A would alter the phase portraits of the center and the improper node to one of the five remaining possibilities. These latter five are the generic phase portraits near zero. The generic picture in the linear case carries over to the nonlinear setting.

7.2. Nonlinear Planar Setting

Consider the nonlinear planar system,

$$\dot{x} = f(x). \tag{7.7}$$

Definition 7.1. *Denote a solution starting at x by $\phi_t^f(x)$. The superscript here keeps track of the system in question. The map $\phi_t^f(x) : \mathbb{R}^2 \to \mathbb{R}^2$ is called the flow map and $\{\phi_t^f(x) : t \in \mathbb{R}\}$ the flow of the system* (7.7). △

Thus,

$$\begin{aligned} \frac{d}{dt}(\phi_t^f(x)) &= f(\phi_t^f(x)) \\ \phi_0^f(x) &= x \qquad \text{(initial condition)}. \end{aligned}$$

Let x_e be an equilibrium point, i.e.

$$f(x_e) = 0.$$

Thus,

$$\phi_t^f(x_e) = x_e \qquad \forall t \in \mathbb{R},$$

(Aside: We recognize that x_e is a fixed point of the flow map.) Let us denote the linearization of the f at x_e by A;

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=x_e}.$$

The solution to the linearization

$$\dot{x} = Ax$$

is given by

$$\phi_t^A(x) = \exp(tA)x.$$

Definition 7.2. *If x_e is an equilibrium such that A has no eigenvalues on the imaginary axis, then we call x_e an hyperbolic equilbrium point.* △

We now are ready for the connection between the linearization and the nonlinear system.

Theorem 7.1 (*Hartman-Grobman*)

Consider the nonlinear system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

with hyberbolic equilibrium x_e. Let $A = (\frac{\partial f}{\partial x})_{x_e}$ denote the linearization of f. Let $\{\phi_t^f\}$ denote the flow of the nonlinear system. Then, there exists a map

$$F : B_\delta \subset \mathbb{R}^n \to \mathbb{R}^n$$

where $B_\delta = \{x : \|x - x_e\| < \delta\}$ is a ball defining a sufficiently small neighborhood of x_e, such that $F(x_e) = 0$, F is one-to-one and onto $F(B_\delta)$, and the map F as well as its inverse F^{-1} are continuous (We call such an F a homeomorphism—see Figure 7.2.), such that,

$$F(\phi_t^f(x)) = \exp(tA)F(x) \qquad x \in B_\delta$$

or more succinctly,

$$\phi_t^f = F^{-1} \circ \exp(tA) \circ F \tag{7.8}$$

□

Remark 7.1. We say that the flow ϕ_t^f is conjugate to the flow $\exp(tA)$. △

Remark 7.2. We have tacitly assumed that the nonlinear system has a well-defined solution for all time. This is not necessarily for the statement of this theorem. We only need existence for $|t| < T$, some $T > 0$. △

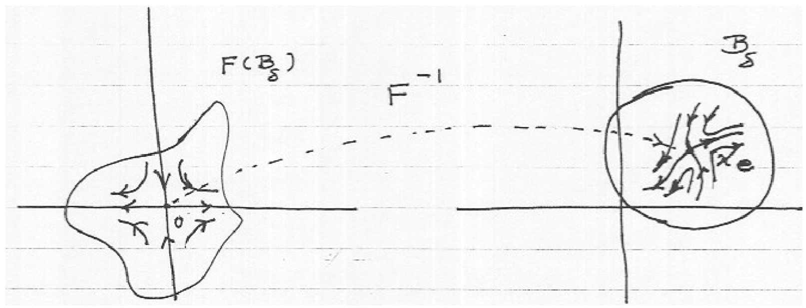


Figure 7.2: Homeomorphism.

As shown in Figure 7.2, the phase portrait of the nonlinear system near an equilibrium x_e is a distorted version (by F^{-1}) of the linearization near zero.

Remark 7.3. The hyperbolicity assumption of the Hartman-Grobman Theorem ensures that the linearization fits into one of the Jordan forms. This assumption is important as the following example illustrates. △

Example 7.1. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2).\end{aligned}$$

The linearization at $(0,0)$ is just the harmonic oscillator with center $(0,0)$. But, from the polar coordinate representation of the system,

$$\begin{aligned}\dot{r} &= -\mu r^3\\ \dot{\theta} &= 1,\end{aligned}$$

so the solutions to the nonlinear system spiral in (out) towards (away from) zero for $\mu > 0$ ($\mu < 0$). The eigenvalues of the linearization at $(0,0)$ lie on the imaginary axis and the equilibrium point is not hyperbolic.

7.3. Closed orbits of a dynamical system

Definition 7.3. *Consider the nonlinear system with flow ϕ_t^f. We say that x is a nontrivial periodic point of period T if $\phi_T^f(x) = x$ for some $T > 0$, where T is the* *smallest such time. The trajectory γ through the periodic point is called a* <u>*periodic orbit*</u>*:*

$$\gamma = \{\phi_t^f(x) : t \geq 0\}.$$

△

Such orbits are closed curves.

Finding periodic orbits is hard. However, in $\mathbb{R}^2$ we have a sufficient condition which we will see in the following theorem (after a brief definition).

Definition 7.4. *A region $M \subset \mathbb{R}^n$ is* <u>*positively (negatively) invariant*</u> *for flow ϕ_t^f, if for each $x \in M$,*

$$\phi_t^f(x) \in M \qquad t \geq 0 \quad (t \leq 0).$$

△

Theorem 7.2 (*Poincaré-Bendixson*)

Consider the continuous time dynamical system <u>*in the plane*</u>

$$\dot{x} = f(x).$$

Let M be a closed and bounded, positively invariant set for the flow $\{\phi_t^f : t \geq 0\}$. Suppose M <u>*does not*</u> *contain any equilibria of the given system. Then M contains a* <u>*closed orbit*</u>*.* □

The following example of a fundamental biochemical process called glycolysis, taken from S. H. Strogatz (Nonlinear Dynamics and Chaos), is a nice illustration of the Poincaré-Bendixson Theorem.

Example 7.2. Glycolysis in living cells generates energy through breaking down sugar. In intact yeast cells, as well as in yeast or muscle extracts, glycolysis proceeds under suitable conditions in an <u>oscillatory</u> fashion with the periodic rise and fall of the concentrations of various intermediates. A simple kinetic model due to Sel'kov (1968) (Eur. J. Biochem. <u>4</u>: 79), in dimensionless form is given by,

$$\begin{aligned}
\dot{x} &= -x + ay + x^2 y \\
\dot{y} &= b - ay - x^2 y
\end{aligned}$$

where x and y are, respectively, the concentration of ADP (adenosine diphosphate) and F6P (frustose-6-phosphate), $a, b > 0$ are kinetic parameters.

To show that this system has a periodic solution via the Poincaré-Bendixson Theorem, one needs to find a positively invariant set for the system, that contains no equilibria, that is closed and bounded. Such a set would also be called a trapping region for the system. There will be conditions on a and b as a result. We construct a trapping region in Figure 7.3.

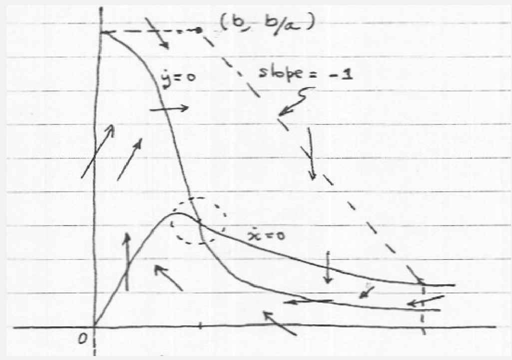


Figure 7.3: Glycolysis Model.

In the figure, the solid curves $\dot{x} = 0$ (equivalently $y = x/(a+x^2)$), and $\dot{y} = 0$ (equivalently $y = b/(a+x^2)$) are the null-clines. On these curves, the direction of the vector fields are marked by vertical and horizontal arrows, respectively. Ignore the dotted circle for the moment. The intersection of the null-clines gives the unique equilibrium point $(b, b/(a+b^2))$. The five-sided figure bounded by the horizontal and vertical axes and the three dotted, straight line segments is a trapping region, in the sense that once a trajectory enters the region, it never leaves it. To see this, convince yourself that the arrows are drawn correctly on the boundaries of the region. (Hint: Verify that in the region above the null-cline $\dot{x} = 0$, we obtain $\dot{x} > 0$ and below it, we obtain $\dot{x} < 0$; in the region to the left of the null-cline $\dot{y} = 0$, but to the right of $x = 0$, $\dot{y} > 0$, and to the right of the null-cline $\dot{y} = 0$ and above $y = 0$, we obtain $\dot{y} < 0$. On the diagonal dotted line of slope -1,

$$\begin{aligned}\dot{x} - (-\dot{y}) &= -x + ay + x^2y + b - ay - x^2y \\ &= b - x,\end{aligned}$$

implying $-\dot{y} > \dot{x}$ if $x > b$ which is the above case.)

We cannot conclude from this analysis that there is a periodic solution in the trapping region. This is because we have an equilibrium point in this region—violating one of the hypotheses of the theorem. What should we do?

Well, if the equilibrium $(b, b/(a+b^2))$ is an unstable node or focus, then on a small dotted circle surrounding the equilibrium, all arrows will be pointing outward. Then one can conclude that the trapping region minus the open disk bounded by the dotted circle, is a closed and bounded, positively invariant set (i.e. a smaller trapping region) containing no equilibria. Hence, it must contain a periodic orbit by Poincaré-Bendixson Theorem.

So, what are the conditions for the equilibrium to be an unstable node or focus?

Linearize the dynamics at the equilibrium to get

$$A = (Df)_{x_e} = \left[\begin{array}{cc} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{array} \right] \Big|_{x=b, y=b/(a+b^2)}$$

Note that the determinant and trace of A give:

$$\det(A) = a + b^2$$
$$\text{tr}(A) = -\frac{b^4 + (2a-1)b^2 + (a+a^2)}{a+b^2}$$

Thus, the equilibrium is unstable if tr$(A) > 0$ and stable if tr$(A) < 0$. The stability regions in parameter space are represented by the curve

$$b^2 = \frac{1}{2}(1 - 2a \pm \sqrt{1-8a})$$

as shown in Figure 7.4.

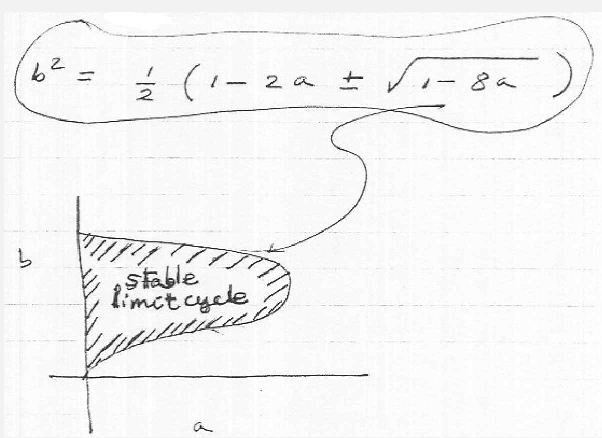


Figure 7.4: Stability Region in Parameter Space.

7.4. Classification of planar linear systems

Recall the classification diagram in parameter space for a planar linear system as shown in Figure 7.5. For the 2×2 matrix $A = [a_{ij}]$, the characteristic polynomial

$$\begin{aligned}\chi_A(s) &= \det \begin{bmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{bmatrix} \\ &= (s - a_{11})(s - a_{22}) - a_{12}a_{21} \\ &= s^2 - s(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \\ &= s^2 - \tau s + \Delta\end{aligned}$$

where $\tau = \mathrm{tr}(A)$ and $\Delta = \det(A)$. The discriminant for the characteristic equation is $\tau^2 - 4\Delta$. Thus, from Figure 7.5, we see that the determinant, trace, and discriminant provide useful information for classifying the linear dynamical systems or equilibrium of linearized systems.

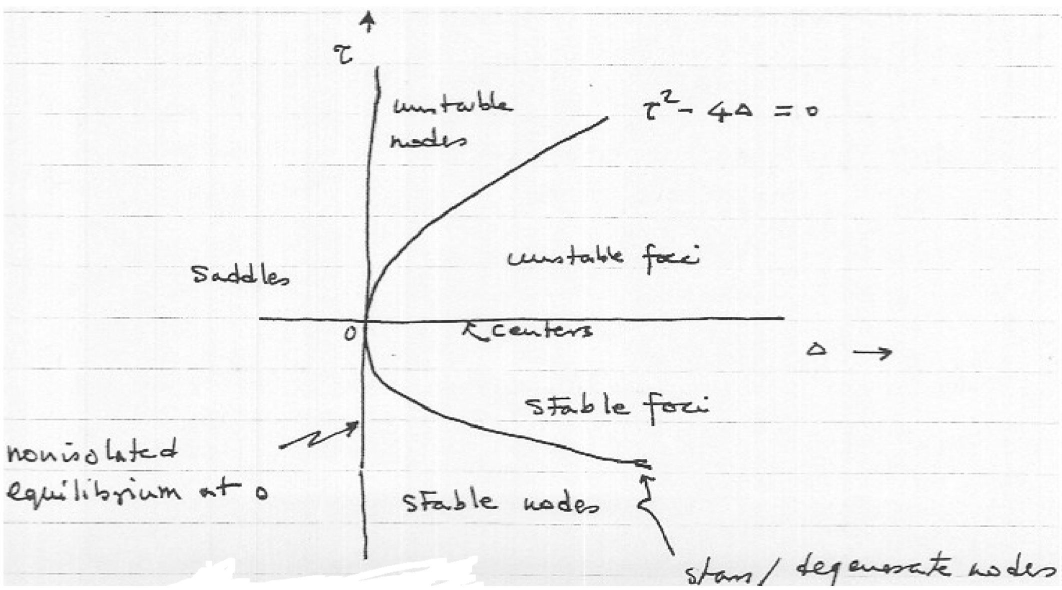


Figure 7.5: Classification diagram for planar linear systems

Remark 7.4. Based on the characteristic polynomial, we see that we cannot have a hyperbolic equilibrium when either $\Delta = 0$ or ($\tau = 0$ and $\Delta > 0$). Review the outcome of exercise 7.2 in light of this claim. $\triangle$

Lecture 8

Index Theory and Introduction to Bifurcations

8.1. Index Theory

We continue our study of ways to recognize the existence (or nonexistence) of periodic solutions to planar nonlinear systems. (Note: All systems we study here are C^1 smooth.)

Remark 8.1. f is C^1 smooth if f_1 and f_2 have continuous first partials. More generally, C^k smooth means f_1 and f_2 have continuous kth partials. $\triangle$

Definition 8.1. *Given a vector field f in the plane and a closed, simple curve γ not passing through an equilibrium point of*

$$\dot{x} = f(x),$$

then the index I_γ^f is simply the total rotation of the vector field as we proceed counterclockwise once around the closed curve γ (see Figure 8.1), measured by

$$I_\gamma^f = \frac{1}{2\pi} \oint_\gamma d\theta_f \tag{8.1}$$

where $\theta_f = \arctan(\frac{f_2}{f_1})$. $\triangle$

The index made its (first) appearance in (H. Poincaré, "Sur les dí finie par les équations différentielles," J. Math. Pures Appl. 4 (1): 167-244, 1885).

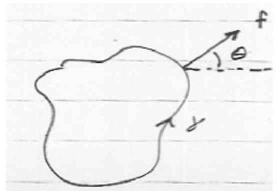


Figure 8.1: Convention for index calculation.

Since $\frac{d}{dz}\arctan(z) = \frac{1}{1+z^2}$, it follows that,

$$I_\gamma^f = \frac{1}{2\pi}\oint_\gamma \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}. \tag{8.2}$$

Property 3 (Homotopy invariance)**.** A key property of the index is that given two curves γ and γ' in the plane such that γ can be continuously deformed in γ' (or homotoped into γ'), without passing through any equilibria, then

$$I_\gamma^f = I_{\gamma'}^f.$$

(Proof: Since I_γ^f is an integer and it varies continuously as γ is being varied continuously, it does not vary at all as long as we do not cross an equilibrium point.) △

Property 4 (Zero index)**.** If γ does not enclose any equilibrium points, $I_\gamma^f = 0$. (Proof: By Property 1, we can shrink γ to a tiny circle without changing the index. But θ_f is essentially constant on such a circle, thanks to the assumed smoothness of the vector field (see Figure 8.2). So, $I_\gamma^f = I_{\gamma'}^f = 0$.) △

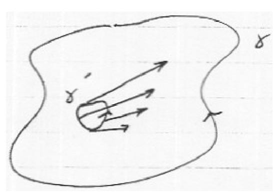


Figure 8.2: Shrinking a simple closed curve not enclosing equilibria.

Property 5.

$$I_\gamma^f = I_\gamma^{-f}$$

(Proof: Use the formula for index.) △

Property 6. If γ is a closed orbit of $\dot{x} = f(x)$, then

$$I_\gamma^f = +1.$$

(Proof: By the assumption that γ is a closed orbit, the vector field is tangential to γ everywhere on γ as in Figure 8.3. Pick a parameterization with respect to t of γ, then use the index formula.) △

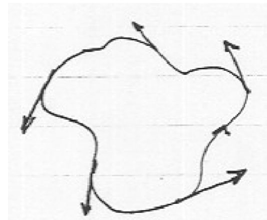


Figure 8.3: Vector field along a periodic trajectory.

From the definition of index, the index of a curve γ when it encloses a single node or focus is $+1$, and it is -1 when it encloses a single saddle. Examine the vector fields in Figure 8.4 to convince yourself.

Exercise 8.1. The index of a center is also $+1$. Why?

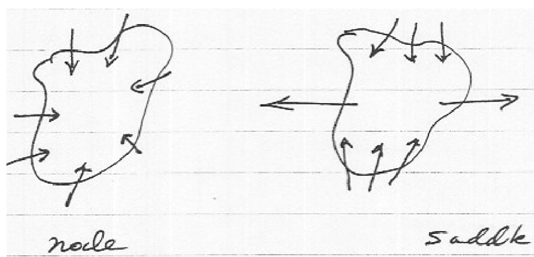


Figure 8.4: Vector fields near a node and a saddle.

Definition 8.2. *The index $I^f(x^*)$ of an isolated equilibrium point x^* is defined to be I^f_γ, where γ is any closed, simple curve enclosing x^* and no other equilibria. (By Property 1 above, this is well-defined, i.e. it is dependent on only x^* and not the particular γ.)* △

Under this definition, we can safely say,

$$I^f(\text{node}) = I^f(\text{focus}) = +1$$
$$I^f(\text{saddle}) = -1$$

Theorem 8.1

If a simple closed curve γ encloses n isolated equilibria $x_1^*, \ldots, x_n^*$ then

$$I^f_\gamma = \sum_{i=1}^{n} I^f(x_i^*). \tag{8.3}$$

□

Proof of Theorem 8.1

Left as an exercise. ■

Exercise 8.2. Prove this theorem by making use of the property of Homotopy invariance.

Corollary 8.1

A periodic orbit must enclose an equilibrium point. □

Exercise 8.3. Does this corollary allow one to rule out oscillation in the glycolysis example when a, b are such that they lie in the unshaded region of the parameter space stability plot?

Example 8.1. Closed orbits are impossible in the population biology model

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where $x, y \geq 0$. This can be shown by the following argument.

Equilibria are $(0, 0)$ (unstable node); $(0, 2)$, $(3, 0)$ (unstable nodes); and $(1, 1)$ (saddle) marked by X's in Figure 8.5.

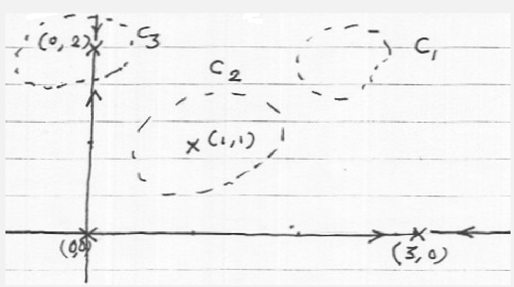


Figure 8.5: Population biology example.

There are three qualitatively distinct possibilities C_i for closed orbits. C_1 and C_2 are ruled out by the theorem. C_3 is also ruled out because the y axis is a stable manifold and trajectories cannot cross.

Another useful result that gives us conditions to exclude periodic orbits from certain regions in the plane is attributed to Bendixson.

Theorem 8.2 (*Bendixson's Criterion*)

Let D be a simply connected region in the plane such that div$(f) \triangleq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero in any subregion of D and also does not change sign in D. Then D does not contain any closed orbits of $\dot{x} = f(x)$. □

Proof of Theorem 8.2

Assume towards contradiction, that γ is a closed orbit in D.

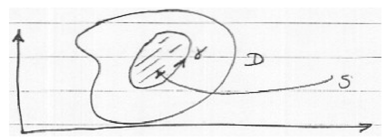


Figure 8.6: Closed orbit in D.

On γ, $\dot{x}_1 = f_1(x_1, x_2)$ and $\dot{x}_2 = f_2(x_1, x_2)$ implies $\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$ or $f_1 dx_2 - f_2 dx_1 \equiv 0$ on γ. Hence,

$$\oint_\gamma f_1 dx_2 - f_2 dx_1 = 0.$$

But by the theorem of Green this implies the surface integral $\int\int$ div$(f) dx_1 dx_2 = 0$, which contradicts the hypothesis that div$(f) \neq 0$ and does not change sign on any subregion of D. Hence there can be no periodic orbits in D. ■

Remark 8.2. We can extend this result as Dulac did—observe that we can multiply by $q(x)$,

$$q f_1 dx_2 - q f_2 dx_1 \equiv 0 \qquad \text{on } \gamma,$$

and hence

$$\oint_\gamma q f_2 dx_1 - q f_1 dx_2 = 0.$$

By Green, this implies, $\int\int$ div$(qf) dx_1 dx_2 = 0$ which would lead to a contradiction if $q(\cdot)$ was picked such that div$(qf) \equiv 0$ on any subregion of D. Note,

$$\text{div}(qf) = \nabla q \cdot f + q\text{div}(f).$$

So, finding $q(x)$ such that div$(qf) = \nabla q \cdot f + qdiv(f) > 0$ (or < 0) on D is a problem of solving a partial differential inequality in q. In practice, one tries to "guess" a $q(x)$. If you can guess a $q(x)$ such that div(qf) is sign-definite, then periodic orbits cannot exist in the region of interest. △

8.2. Brief Introduction to Bifurcations

When a differential equation has a settable parameter, the study of how the phase-portrait changes as one continuously varies the parameter is referred to as bifurcation theory. The term originates in the branching of equilibria as in the Euler buckling problem. We start in one dimension.

For $\dot{x} = r + x^2$ the phase portrait for different values of r can be 'stacked' in one figure as shown in Figure 8.7.

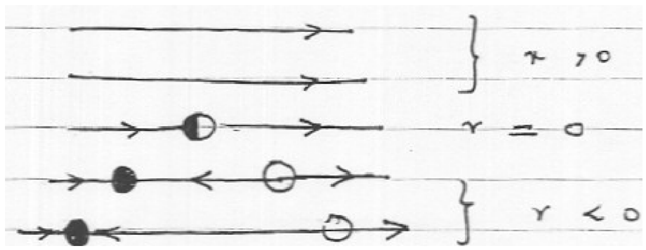


Figure 8.7: 'Stacked' phase portraits for a scalar ODE.

The filled circle indicates a stable equilibrium at $x_e = -\sqrt{-r}$ (for $r < 0$), while the open circle indicates an unstable equilibrium at $x_e = \sqrt{-r}$ (for $r < 0$). The half-filled circle at $x_e = 0$ (for $r = 0$) is a "half-stable" equilibrium as the arrows suggest.

Rotating Figure 8.7 $90°$ degrees clockwise and flipping the x axis, we get the bifurcation diagram shown in Figure 8.8.

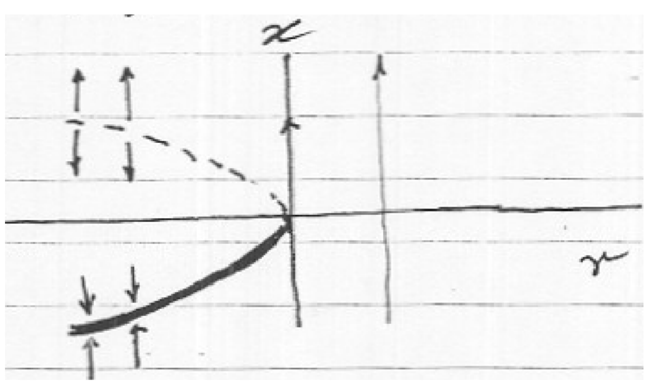


Figure 8.8: Bifurcation diagram for $\dot{x} = r + x^2$.

The dotted line indicates an unstable branch (of equilibria) and the solid line, a stable branch. This type of bifurcation is referred to as a saddle-node (turning point / fold / or blue sky) bifurcation.

If we had used the form $\dot{x} = r - x^2$, the picture would look like Figure 8.9.

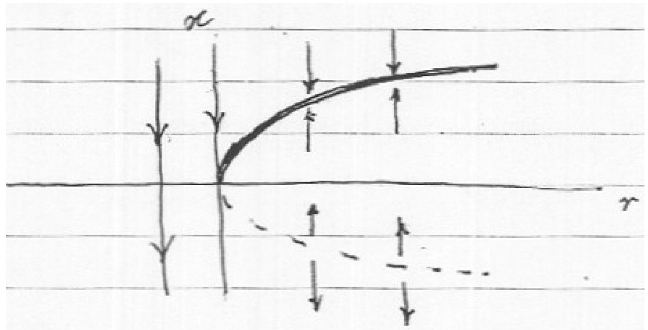


Figure 8.9: Bifurcation diagram for $\dot{x} = r - x^2$.

As r increases beyond zero, two branches of equilibria appear ("out of the blue sky"). The appearance or disappearance of branches has to do with occurrence of a singularity in $\left(\frac{\partial f}{\partial x}\right)\Big|_{x_e}$ for a critical value of r.

One can also write down forms that are transcritical (i.e. no change in the number of branches). Consider $\dot{x} = rx - x^2$. Then $x_e = 0$ is always an equilibrium, so is $x_e = r$. The bifurcation diagram is shown in Figure 8.10.

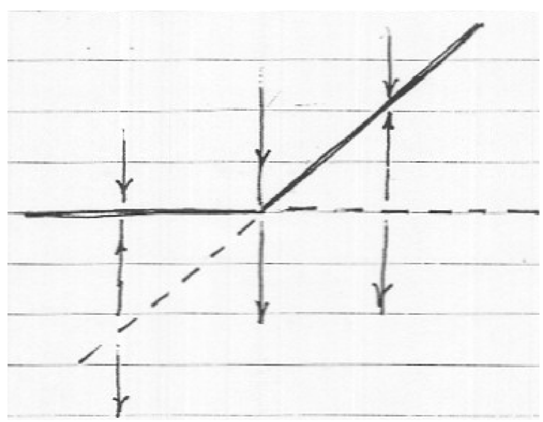


Figure 8.10: Bifurcation diagram for $\dot{x} = rx - x^2$.

Figure 8.10 illustrates that in a transcritical bifurcation there is an exchange of stability, as shown with the exchange of stability between the two branches.

When the order of the right-hand side increases, additional branches appear. For example, $\dot{x} = rx - x^3$ (invariant under $x \to -x$) yields the aptly named supercritical pitchfork bifurcation.

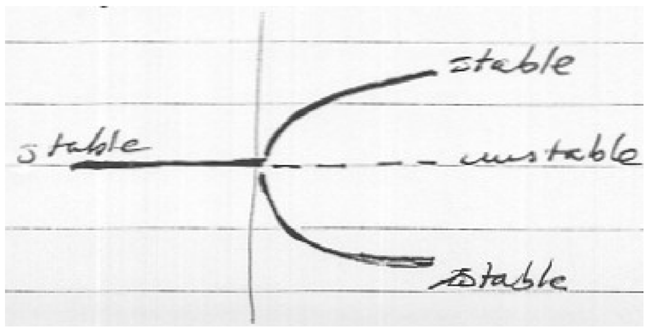


Figure 8.11: Bifurcation diagram for $\dot{x} = rx - x^3$.

Consider $\dot{x} = rx + x^3 - x^5$ with bifurcation diagram (subcritical pitchfork) shown in Figure 8.12. This suggests possibilities of hysteretic jumps between the $x_e = 0$ stable branch and the nontrivial stable branches.

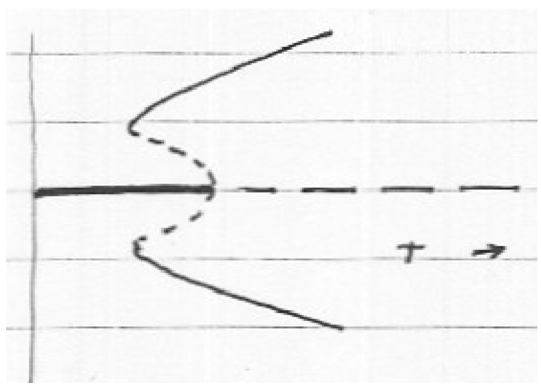


Figure 8.12: Bifurcation diagram for $\dot{x} = rx + x^3 - x^5$.

One can embed all these normal forms in two dimensions. As we see illustrated by the system

$$\dot{x} = r + x^2$$
$$\dot{y} = -\lambda y$$

Suppose $\lambda > 0$. Then for $r < 0$, the branch ($x_e = -\sqrt{-r}$, $y_e = 0$) is a branch of (stable) nodes, while the branch ($x_e = \sqrt{-r}$, $y_e = 0$) is a branch of saddles. This is the origin of the term saddle-node bifurcation.

We postpone for now the discussion of dynamic bifurcations such as the Hopf bifurcation. This is the setting in which stable limit cycles emerge when an equilibrium at zero loses stability.

Lecture 9

Stability Theory: Autonomous Systems - Part I

The principles of mechanics as developed in the works of Laplace, Lagrange, and Dirichlet lead to tools for understanding the stability properties of solutions of nonlinear systems. The modern period in this direction begins with the classical memoir of A. M. Lyapunov (also, Liapunov and Liapounoff) (Probleme générale de la stabilité de mouvement, Ann. Fac. Sci. Toulose, 9: 203:474 (1907)—translation of a paper published in Russian in Comm. Soc. Math., Kharkov 1892, facsimile reproduction as Annals of Mathematics Study No. 17 (Princeton Univ. Press), 1947). The key ingredients are: (a) a definition of stability, and (b) an "energy" method to assess stability. We introduce these in the setting of autonomous differential equations.

An equilibrium point x_e of a system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n \tag{9.1}$$

satisfies $f(x_e) = 0$. Let $y = x - x_e$ and define $g(y) = f(y + x_e)$. Then the equilibrium point $y = 0$ of the system

$$\dot{y} = g(y) \tag{9.2}$$

corresponds to the equilibrium point x_e of the system (9.1). Thus the shift of the origin in $\mathbb{R}^n$ allows one to refer to an equilibrium at zero. This is a standard device in most treatments—we don't use it below.

Definition 9.1 (Lyapunov Stability)**.**

(i) The equilibrium x_e of (9.1) *is said to be stable in the sense of Lyapunov, if given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x_0 \in B_\delta(x_e)$$

implies that the solution starting at x_0, denoted $x(t)$, is trapped in $B_\epsilon(x_e)$:

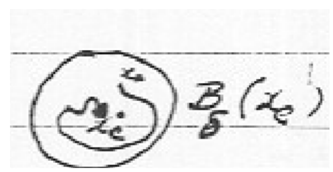
$$x(t) \in B_\epsilon(x_e) \qquad \forall t \geq 0.$$

Here $B_r(z)$ stands for the open ball of radius r centered at z: $B_r(z) = \{x : \|x - z\| < r\}$ with respect to a choice of norm $\|\cdot\|$ in $\mathbb{R}^n$.

(ii) *x_e is unstable if it is not stable.*

(iii) *x_e is asymptotically stable if it is stable as in (i), and $\delta > 0$ can be chosen such that $\lim_{t\to\infty} x(t) = x_e$ for every initial condition $x_0 \in B_\delta(x_e)$ (This is attractivity of x_e).*

(a) Figure for stability in the sense of Lyapunov.



(b) Figure for asymptotic stability.

Figure 9.1: Illustrations for Lyapunov stability theory.

△

The basic theorem of the subject, due to Lyapunov, gives a sufficient condition for stability (or asymptotic stability) of an equilibrium point x_e.

Theorem 9.1 (*Lyapunov*)

Let x_e be an equilibrium point of the system (9.1) satisfying the local Lipschitz condition. Let D be a domain (open, connected region) of $\mathbb{R}^n$, containing x_e. Suppose $V : D \to \mathbb{R}$ is a C^1 function and

$$\begin{aligned} V(x_e) &= 0 \\ V(x) &> 0 \qquad \text{in} \quad D - \{x_e\} \\ \dot{V}(x) &\leq 0 \qquad \text{in} \quad D. \end{aligned}$$

Then, x_e is stable. Moreover, if $\dot{V} < 0$ in $D - \{x_e\}$, then x_e is asymptotically stable. □

Proof of Theorem 9.1

Refer to Figure 9.2 for illustrations of the nested sets used throughout this proof.

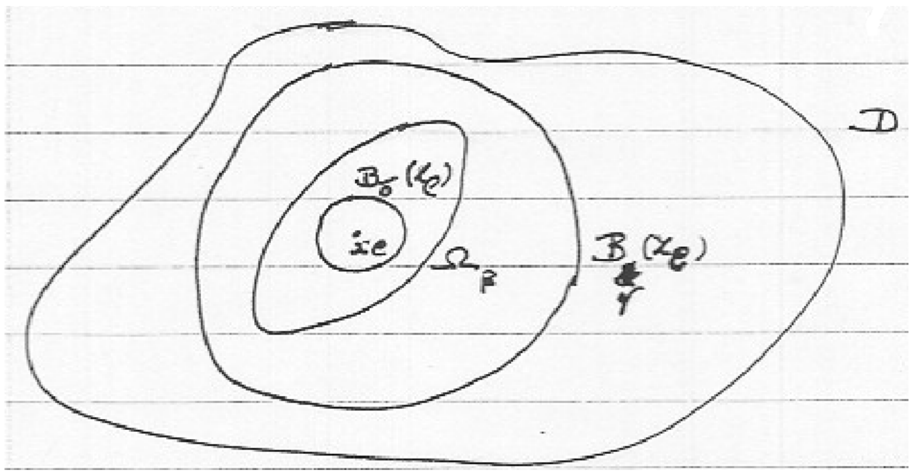


Figure 9.2: Nested sets.

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that we have closed ball $\bar{B}_r(x_e) = \{x : \|x - x_e\| \leq r\} \subset D$. Let $\alpha = \min_{\|x - x_e\| = r} V(x)$. Note that $\alpha > 0$ by hypothesis. Take β such that $0 < \beta < \alpha$, and let

$$\Omega_\beta = \{x \in \bar{B}_r(x_e) : V(x) \leq \beta\}.$$

Then $\Omega_\beta \subset B_r(x_e)$ (Proof: Suppose not. Let $p \in \Omega_\beta$ be such that $\|p - x_e\| = r$. Then $V(p) \geq \alpha > \beta$, a contradiction).
Let x(t) be a solution with $x(0) \in \Omega_\beta$. Since $\dot{V}(x(t)) \leq 0$, $V(x(t)) \leq V(x(0))$ and hence $x_e \in \Omega_\beta$, $\forall t \geq 0$. Since Ω_β is a closed and bounded set, we conclude (by the Long Time Existence Uniqueness Theorem previously discussed) that we have a unique solution within Ω_β, a positively invariant set, for all time.
V is continuous at x_e and $V(x_e) = 0$ implies there is a $\delta > 0$ such that

$$\|x - x_e\| \leq \delta \implies V(x) < \beta.$$

Thus,

$$B_\delta(x_e) \subset \Omega_\beta \subset B_r(x_e),$$

and

$$\begin{aligned} x(0) \in B_\delta(x_e) &\implies x(0) \in \Omega_\beta \\ &\implies x(t) \in \Omega_\beta \qquad \forall t \geq 0 \\ &\implies x(t) \in B_r(x_e) \qquad \forall t \geq 0 \\ &\implies x(t) \in B_\epsilon(x_e) \qquad \forall t, r \geq 0 \qquad (\text{since } \epsilon \geq r). \end{aligned}$$

Under the extra assumption $\dot{V} < 0$ in $D - \{x_e\}$, one can show that the monotone decreasing function $V(x(t))$ has a limit $c = 0$. (Here we are appealing to the 'well-ordering' of $\mathbb{R}$.) The existence of a limit $c \geq 0$ is assured by the *lower bound on V. Suppose $c > 0$. By continuity of V there exists $d > 0$ such that*

$$B_d(x_e) \subset \Omega_c = \{x \in \bar{B}_r(x_e) : V(x) \leq c\}.$$

Since $\lim_{t\to\infty} V(x(t)) = c > 0$ by hypothesis, $x(t)$ never enters $B_d(x_e)$. Let

$$-\gamma = \max_{d \leq \|x - x_e\| < r} \dot{V}(x).$$

By hypothesis, $\gamma > 0$. By the fundamental theorem of integral calculus,

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ &\leq V(x(0)) - \gamma t \end{aligned}$$

Since the right-hand side eventually becomes negative, we get a contradiction from assuming $c > 0$. So, $c = 0$. Hence,

$$\lim_{t\to\infty} V(x(t)) = 0.$$

By hypothesis on V,

$$\lim_{t\to\infty} x(t) = x_e.$$

■

Remark 9.1. The above result is a prototype stability theorem in the spirit of the energy method in mechanics (more on this later). In the abstract setting of ordinary differential equations (not necessarily hamiltonian or dissipative) the Lyapunov function V is a stand-in for energy functions from mechanics. △

Note that for the system $\dot{x} = f(x)$, the time derivative of a Lyapunov function along trajectories of the system may be written

$$\dot{V}(x) = \nabla V(x) \cdot f \tag{9.3}$$

$$= \frac{\partial V}{\partial x} \cdot f. \tag{9.4}$$

Suppose the state space can be factored into position and momentum variables, $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$, and

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned} \tag{9.5}$$

Then,

$$\frac{dH}{dt} = \frac{\partial H}{\partial q} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(-\frac{\partial H}{\partial q}\right) = 0$$

If we further assume that $x_e = (q_e, p_e)$ is an equilibrium with $H(x) > H(x_e)$ for all $x \in D$ a neighborhood of x_e, then

$$V(x) = H(x) - H(x_e) \qquad \text{and} \qquad \dot{V}(x) = \frac{dH}{dt} = 0$$

satisfies the hypotheses of Lyapunov's theorem allowing one to conclude that x_e is a stable equilibrium.

Suppose $H(x)$ is of the form kinetic-plus-potential energy,

$$H(q,p) = \frac{1}{2} p \cdot M^{-1} p + V(q),$$

where $M = M^T > 0$ is a constant mass matrix and $V(q)$ is a potential. Then (q_e, p_e) is an equilibrium of the dynamics

$$\begin{aligned} \dot{q} &= M^{-1} p \\ \dot{p} &= -\frac{\partial V}{\partial q} \end{aligned} \tag{9.6}$$

if and only if $p_e = 0$ and q_e is a critical point of V. If further q_e is an isolated/strict local minimum of V, then (q_e, p_e) is a strict local minimum of H and hence a stable equilibrium.

The result just derived is known as the Lagrange-Dirichlet theorem and is a guiding principle in much of mechanics. It asserts the proper role of (potential) energy minimization in stability in the correct dynamical sense of Lyapunov, superseding earlier quasi-static notions (e.g. due to Torricelli).

Corollary 9.1

If x_e is a (stable) equilibrium of a nontrivial hamiltonian system (9.5), *it can never be asymptotically stable.* □

Proof of Corollary 9.1

There exists $x_0 \in D$, $H(x_0) \neq H(x_e)$. For any trajectory beginning at x_0, $H(x(t)) = H(x(0)) \neq H(x_e)$. Hence, by continuity of H, $\lim_{t\to\infty} x(t) \neq x_e$, even if such a limit exists. ■

Hamiltonian systems of the form of (9.6) are said to be natural mechanical systems. A vast array of systems in classical physics, molecular dynamics, and engineering take this form, or its dissipative modification

$$\begin{aligned} \dot{q} &= M^{-1} p \\ \dot{p} &= -\frac{\partial V}{\partial q} - R(q)\dot{q} \end{aligned} \tag{9.7}$$

where $R(q) = R^T(q) > 0$ defines the Rayleigh dissipation function

$$\mathfrak{R}(q, \dot{q}) = \frac{1}{2}\dot{q} \cdot R(q)\dot{q}.$$

Along trajectories of (9.7),

$$\begin{aligned} \frac{dH}{dt} &= -M^{-1}p \cdot (R(q)M^{-1}p) \\ &\leq 0 \qquad \text{(by hypothesis)}. \end{aligned}$$

Suppose D is such that $(q_e, 0)$ is the only equilibrium in D (thus q_e is a critical point of V), and q_e is a strict local minimum of V. Then, can we prove asymptotic stability of $(q_e, 0)$? While stability is assured by the Lagrange-Dirichlet theorem, one cannot use H as the Lyapunov function for the asymptotic stability argument. This is because,

$$\left.\frac{dH}{dt}\right|_{(q,0)} = 0.$$

Figure 9.3 illustrates this problem.

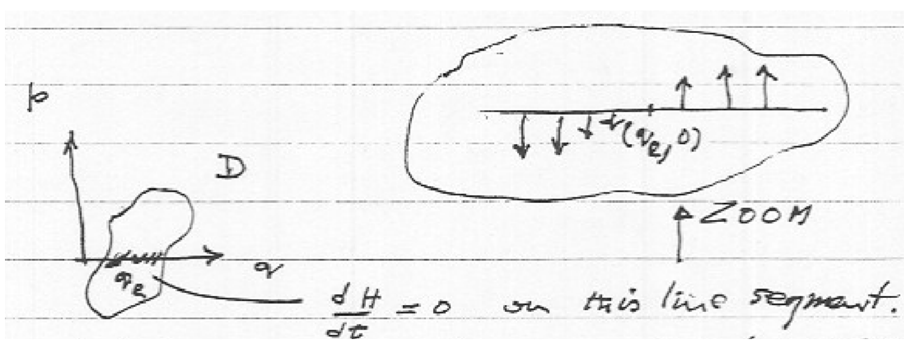


Figure 9.3: $\frac{dH}{dt} = 0$ along $(q, 0)$.

None of the points on the line segment is an equilibrium (except for $(q_e, 0)$) by hypothesis on D. Does this means that we get convergence to $(q_e, 0)$ anyway? One needs a new idea—the invariance principle of LaSalle, if one insists on working with H as a possible Lyapunov function. Another alternate is to "fix up" the Lyapunov function, i.e. add suitable extra terms to H. Both of these approaches are important in solving a range of problems and we will discuss both.

Example 9.1 (Fixing up a Lyapunov function for a damped pendulum)**.** In equation (9.7), consider $n = 1$, $M = 1$ constant, $V(q) = \frac{g}{l}(1 - \cos(q))$, $R(q) =$

$b > 0$. Then,

$$\begin{aligned}\dot{q} &= p\\ \dot{p} &= \frac{-g}{l}\sin(q) - bp\\ H(q,p) &= \frac{p^2}{2} + gl(1-\cos(q))\\ \frac{dH}{dt} &= -bp^2 \qquad \text{(where } (q,p)=(0,0) \text{ is an equilibrium).}\end{aligned}$$

Define

$$\tilde{H} = \frac{1}{2}[q,p]\begin{bmatrix} a_{11} & a_{12}\\ a_{12} & a_{22}\end{bmatrix}\begin{bmatrix} q\\ p\end{bmatrix} + \frac{g}{l}(1-cos(q)).$$

Note: we require $a_{12} = a_{21}$ so this matrix is symmetric. We seek $[a_{ij}] > 0$ such that

$$\begin{aligned}\frac{d\tilde{H}}{dt} &= (a_{11}q + a_{12}p + \frac{g}{l}\sin(q))p + (a_{12}q + a_{22}p)(-\frac{g}{l}\sin(q) - bp)\\ &= \frac{g}{l}(1-a_{22})p\sin(q) - \frac{g}{l}a_{12}q\sin(q) + (a_{11} - a_{12}b)pq + (a_{12} - a_{22}b)p^2\\ &< 0 \qquad \text{(on a suitable } D\text{).}\end{aligned}$$

Pick $a_{22} = 1, a_{11} = a_{12}b$, <u>and</u> $0 < a_{12} < b$. Then,

$$\begin{bmatrix} a_{11} & a_{12}\\ a_{12} & a_{22}\end{bmatrix} = \begin{bmatrix} a_{12}b & a_{12}\\ a_{12} & 1\end{bmatrix} > 0.$$

Further, we have

$$\begin{aligned}\frac{d\tilde{H}}{dt} &= -\frac{g}{l}a_{12}q\sin(q) - (b - a_{12})p^2\\ &< 0 \qquad \text{on } D = \{(q,p) : |q| < \pi\} - \{(0,0)\}.\end{aligned}$$

Thus, $\tilde{H}$ is a Lyapunov function asserting asymptotic stability of $(0,0)$.

$$\begin{aligned}\tilde{H} &= H + \frac{a_{12}b}{2}q^2 + a_{12}pq\\ &= H + \text{“fixup term”}.\end{aligned}$$

Example 9.2 (Gradient dynamics)**.** Let $V : \mathbb{R}^n \to \mathbb{R}$ and

$$\dot{x} = -\nabla V(x) \qquad x \in \mathbb{R}^n. \tag{9.8}$$

x_e is an equilibrium if and only if it is a critical point of V. Suppose it is an isolated local minimum of V. From

$$\frac{dV}{dt} = -\nabla V \cdot \nabla V < 0$$

for all $x \in B_\epsilon(x_e) - \{x_e\}$ for $\epsilon > 0$ small enough, we conclude asymptotic stability of x_e.

9.1. The Invariance Principle

The Invariance Principle of LaSalle is based on a fundamental result of G. D. Birkhoff (1884-1944), an American mathematician of the first half of the 20th century and a founder of the modern theory of dynamical systems.

Theorem 9.2 (*Birkhoff*)

If a trajectory $x(t)$ of a dynamical system is bounded then

$$L_+ = \omega\text{-limit set of } \{x(t) : t \geq 0\}$$

is a nonempty, compact, invariant set. (Note: on $\mathbb{R}^n$ with metric $d(\cdot,\cdot)$, compact $\iff$ closed and bounded).

Moreover, $x(t) \to L_+$ as $t \to \infty$ in the sense that

$$\begin{aligned}\lim_{t\to\infty} d(x(t), L_+) &= \lim_{t\to\infty} \min_{p\in L_+} d(x(t), p)\\ &= 0.\end{aligned}$$

□

Proof of Theorem 9.2

We omit the proof of Birkhoff's theorem (see for instance Khalil Appendix A.2). ■

Theorem 9.3 (*LaSalle*)

Let Ω be compact (closed and bounded) set with the property that $x(t) \in \Omega$, $\forall t \geq 0$, whenever $x(0) \in \Omega$.

Let $V : \Omega \to \mathbb{R}$ be a C^1 function such that $\dot{V} \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω tends to M at $t \to \infty$. □

Remark 9.2. We call such a V a LaSalle function. △

Proof of Theorem 9.3

Let $x(t)$ be a solution such that $x(t) \in \Omega$, $\forall t \geq 0$. Since $\dot{V} \leq 0$ in Ω, $V(x(t))$ is a monotone decreasing function of t. Since $V(x(t))$ is continuous in the compact *set Ω, it is bounded below on Ω. Therefore $V(x(t)) \to a$ as $t \to \infty$. $L_+ \subset \Omega$ since Ω closed. For any $p \in L_+$ there is a sequence $t_1 \leq t_2 \leq \ldots \leq t_n \leq \ldots$ with $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} x(t_n) = p$.*

By continuity of V,

$$V(p) = \lim_{n\to\infty} V(x(t_n)) = a.$$

Hence, $V(x) = a$ on L_+. Since L_+ is invariant (Birkhoff), $\dot{V}(x) = 0$ on L_+. Since M is the largest invariant set $\subset E$, we get

$$L_+ \subset M \subset E \subset \Omega.$$

Since $\{x(t)\}_{t\geq 0}$ is bounded, $x(t) \to L_+$ as $t \to \infty$.

Hence, $x(t) \to M$ as $t \to \infty$. ■

Remark 9.3. In many problems M is much easier to determine than L_+. △

In some problems, one can pick Ω and V such that, x_e = equilibrium $\in \Omega$ and in fact $M = \{x_e\}$. If further x_e is stable, it is asymptotically stable by LaSalle's theorem.

Revisiting equation (9.7) we can take H as the LaSalle function and use invariance to prove asymptotic stability instead of fixing up the Lyapunov function, as previously done. We take

$$\Omega_c = \{(q,p) : H(q,p) \leq c\}.$$

If we choose $c > 0$ such that $H(q_e, 0) \leq c$, and c is small enough, Ω_c is closed and bounded. Then we also have that

$$\begin{aligned} E &= \{(q,p) : -M^{-1}p \cdot R(q)M^{-1}p = 0\} \subset \Omega_c \\ &= \{(q,p) \in \Omega_c : p = 0\} \end{aligned}$$

We now derive the condition for membership in M, the largest invariant set in E. From the dynamics, we have

$$\dot{q} \equiv 0 \implies q(t) \equiv q_e \qquad \text{(a constant)}$$

and

$$\dot{p} \equiv 0 \equiv -\frac{\partial V}{\partial q}\bigg|_{q_e} - R(q)(0) \implies \frac{\partial V}{\partial q}\bigg|_{q_e} = 0.$$

Thus, we have shown that $(q,p) \in M$ implies $p = 0$ and q_e is a critical point of V. Choosing c small enough, we can ensure $\{(q_e, 0)\} = M$ where q_e is a local minimum of V. Then, we have shown that all trajectories starting in $\Omega_c \to (q_e, 0)$.

Lecture 10

Stability Theory: Autonomous Systems - Part II

10.1. Region of Attraction

Definition 10.1. *For the nonlinear system $\dot{x} = f(x)$, with equilibrium x_e, we define the region of attraction of x_e denoted $\mathcal{R}_A(x_e)$ to be the set*

$$\mathcal{R}_A(x_e) = \{x_0 : \phi_t^f(x_0) \to x_e \text{ as } t \to \infty\}. \tag{10.1}$$

△

If x_e is an asymptotically stable equilibrium, then one can prove $\mathcal{R}_A(x_e)$ is an open, invariant set containing x_e. When is $\mathcal{R}_A(x_e)$ the whole state space? Clearly, if $\mathcal{R}_A(x_e) = \mathbb{R}^n$, then x_e is the unique equilibrium. Additionally, one can rule out all other invariant sets (e.g. periodic orbits). The following theorem provides the set of sufficient conditions.

Theorem 10.1 (*Barabashin-Krasovskii*)

Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that

(i) $V(0) = 0, \qquad V(x) > 0 \qquad x \neq 0,$

(ii) $V(x) \to \infty$ as $\|x\| \to \infty \qquad$ (radial unboundedness),

(iii) $\dot{V}(x) < 0 \qquad \forall x \neq 0.$

Then, $x = 0$ is globally asymptotically stable ($\implies \mathcal{R}_A(0) = \mathbb{R}^n$). □

Remark 10.1. There is no loss of generality in picking $x_e = 0$ here—recall the change of coordinates trick. △

Proof of Theorem 10.1

Let $p \in \mathbb{R}^n$, $V(q) = c > 0$. From (ii), there exists $r > 0$ such that

$$\{x \in \mathbb{R}^n : V(x) \leq c\} = \Omega_c \subset B_r.$$

Thus Ω_c is closed and bounded. By LaSalle's invariance principle, $x(t)$ a trajectory beginning in Ω_c goes to M, the largest invariant set of $E = \{x : \dot{V}(x) = 0\}$ as $t \to \infty$.

By (iii), $M = \{0\}$. Thus, 0 is asymptotically stable and globally attractive. ■

10.2. Instability

The idea of using "energy-like" functions to prove instability and a fundamental result based on this idea can be attributed to Nikolai Gueryevich Chetaev (also Cetaev or Chetayev), a Soviet mechanician, who held the chair of Theoretical Mechanics at Moscow State University.

Theorem 10.2 (*Chetaev's Instability Theorem*)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1 function defined on an open, connected subset D of $\mathbb{R}^n$ which contains 0 with $V(0) = 0$. Choose r such that $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset D$. If,

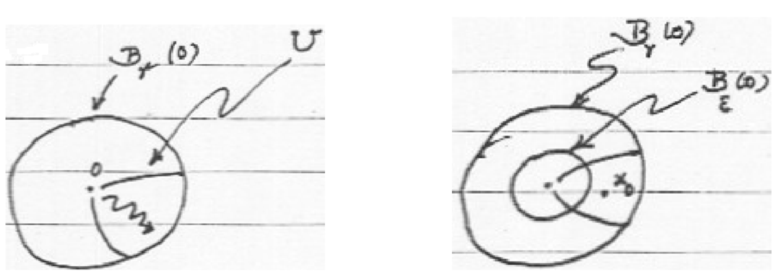
(i) for every $\epsilon > 0$, there exists $x_0 \in B_\epsilon(0)$ such that $V(x_0) > 0$, and

(ii) for $U = \{x \in B_r(0) : V(x) > 0\}$ and $\dot{V}(x) \geq \phi(\|x\|) > 0$ on U, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing with $\phi(0) = 0$,

then, $x = 0$ is an unstable equilibrium. □

Remark 10.2. Such a function ϕ is called a Class K function. △

Proof of Theorem 10.2



(a) (b)

Figure 10.1: Illustrations for proof of Chetaev's instability theorem.

Let $x_0 \in$ *int*(U) *and* $V(x_0) = a > 0$. *Suppose* $\|x_0\| = r_0$. *Consider the annulus* $\{x : \epsilon \le \|x\| \le r\}$ *where* $\epsilon > 0$ *is such that* $V(x) \le \frac{a}{2}$ *for* $\|x\| \le \epsilon$. *Such an* ϵ *exists by continuity of* V *at the origin. Then, since* $\dot{V}(x(t)) > 0$ *and* $V(x_0) = a$, *it follows that a trajectory cannot enter* B_ϵ.

The annulus $\{x : \epsilon \le \|x\| \le r\}$ *is a closed and bounded set and* $\dot{V}(x) > 0$ *in* U. *Since* $\dot{V} \ge \phi(\|x\|)$, *a* *Class* $\mathcal{K}$ *function,* $\dot{V} \ge \gamma > 0$ *for some* γ. *We can take*

$$\gamma = \inf\{\phi(\lambda) : \lambda \in [\epsilon, r]\}.$$

Then,

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(s))ds \\ &\ge a + \int_0^t \gamma ds \\ &= a + \gamma t. \end{aligned}$$

But $V(\cdot)$ *is bounded in* U. *Thus,* $x(t)$ *cannot remain forever in* U. *In fact it leaves* U *at the latest by*

$$T = \frac{1}{\gamma}\left(\sup_U V - a\right).$$

Now $x(t)$ *cannot leave* U *through the surface* $\{x : V(x) = 0\}$. *since* $V(x(t)) \ge a > 0$. *Hence it must leave through the sphere at* $\|x\| = r$. *Since this can happen for arbitrarily small* $\|x_0\|$, *the origin is unstable.* ■

Remark 10.3. We call such a V a Chetaev function. △

Example 10.1. Consider the planar system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $|g_i(x)| \le k\,\|x\|_2^2$ in a neighborhood D of zero ($\implies$ $g_i(0) = 0$ and thus zero is an equilibrium point). Let $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. On the line $x_2 = 0$, i.e. the x_1 axis, $V(x) > 0$ at points arbitrarily close to the origin. Pick

$$U = \{(x_1, x_2) \in B_r(0) : V(x_1, x_2) > 0\},$$

i.e. the shaded area in Figure 10.2.

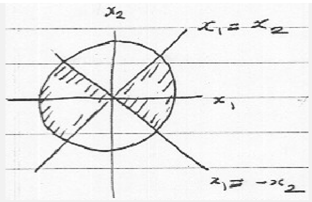


Figure 10.2: Illustration for region U.

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x).$$

But

$$\begin{aligned}|x_1 g_1(x) - x_2 g_2(x)| &\le \sum_{i=1}^{2} |x_i||g_i(x)| \\ &\le 2k \|x\|_2^3 \qquad \text{(by hypothesis)}.\end{aligned}$$

Hence,

$$\begin{aligned}\dot{V}(x) &\ge x_1^2 + x_2^2 - |x_1 g_1(x) - x_2 g_2(x)| \\ &\ge x_1^2 + x_2^2 - 2k \|x\|_2^3 \\ &= \|x\|_2^2 (1 - 2k \|x\|_2).\end{aligned}$$

Choose $r > 0$ such that $B_r(0) \subset D$ and $r < \frac{1}{2k}$. Then $\dot{V}(x) > 0$ on U.

By Chetaev's theorem the origin is unstable. (Question: What is ϕ here?)

Remark 10.4. Note in the previous example how the Chetaev function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ creates a convenient sector for which $V(x) > 0$. △

Example 10.2.

$$M\ddot{x} + (S + \epsilon R)\dot{x} + Kx = 0$$

$M = M^T > 0$, $S = -S^T$, $R = R^T > 0$, $K = K^T$ indefinite, and $\epsilon > 0$. Let $p = M\dot{x}$. Then

$$\begin{aligned}\dot{x} &= M^{-1} p \\ \dot{p} &= -(S + \epsilon R) M^{-1} p - Kx\end{aligned}$$

The hamiltonian is given by

$$H = \frac{1}{2}p^T M^{-1} p + \frac{1}{2}x^T K x.$$

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p} \\ &= Kx \cdot M^{-1}x + M^{-1}p \cdot ((-S - \epsilon R)M^{-1}p - Kx) \\ &= -\epsilon p \cdot M^{-1}RM^{-1}p \qquad (\text{since } S = -S^T).\end{aligned}$$

Pick the Chetaev function to be $V = -H$. Convince yourself by arguing as in the previous example (that is, by picking U properly), that $(0, 0)$ is unstable in $\mathbb{R}^n \times \mathbb{R}^n$.

There are instability results due to Lyapunov that precede the work of Chetaev and were motivating influences on Chetaev's work.

Theorem 10.3 (*Lyapunov - Instability I*)

If V in C^1 in a domain $D \subset \mathbb{R}^n$ with $0 \in D$, $f(0) = 0$ and $V(0) = 0$. Also, $\dot{V} > 0$ when $x \neq 0$ in D, and $V(x)$ assumes positive values arbitrarily near zero, then zero is an unstable equilibrium. □

Proof of Theorem 10.3

Without loss of generality, let D be a bounded domain. V is bounded in D. Let $r \leq R$, $B_r(0) \subset D$. Then $\exists x_0 \in B_r(0)$ such that $V(x_0) > 0$.

Since $\dot{V} > 0$, $V(x(t))$ can only increase, $x(t)$ does not go to zero. In fact, since $\dot{V} \geq m > 0$, $x(t)$ does not even go to a fixed point in $B_R(0)$. $V(x(t))$ must increase indefinitely and hence $x(t)$ must eventually leave $B_R(0)$ via some point in the boundary $\partial B_R(0)$. ■

Theorem 10.4 (*Lyapunov - Instability II*)

Same hypothesis on V as in the previous theorem above, and let $\dot{V} = \lambda V + V^*$ where $V^* > 0$ in D and $\lambda > 0$. Then zero is an instable equilibrium. □

Proof of Theorem 10.4

Let $x_0 \in B_r(x_0)$ as in previous proof. $V(x_0) > 0$.

$$\dot{V} = \lambda V + V^* \implies$$

$$\frac{d}{dt}(\exp(-\lambda t)V) = \exp(-\lambda t)V^* \geq 0.$$

Hence, along $x(t)$, $V \geq \exp(\lambda t)V(x_0)$. V increases indefinitely along $x(t)$, which implies instability. ■

10.3. Some special Lyapunov functions

Let $V = V_p(x) + V_{p+1}(x) + \dots$ be defined in a neighborhood of zero, where V_k is a homogeneous polynomial of degree k. Then, the sign of V in a suitable neighborhood Ω of the origin is the same as the sign of V_p.

Lemma 10.1

If p is odd, V cannot be a Lyapunov function. □

Proof of Lemma 10.1

Let u_i denote the ratio of $\frac{x_i}{x_n}$. Then each x_i can be expressed as

$$\begin{aligned} x_1 &= x_n u_1 \\ x_2 &= x_n u_2 \\ &\vdots \\ x_{n-1} &= x_n u_{n-1} \\ x_n &= x_n \cdot 1. \end{aligned}$$

This implies that

$$V_p = x_n^p V_p(u_1, u_2, \dots, u_{n-1}, 1).$$

Keeping the u_i fixed, the sign of V_p will be the sign of x_n^p or $-x_n^p$ (one of the two, but not both). Since p is odd, x_n^p, $-x_n^p$ may assume both positive and negative values near zero, so V is not positive definite. We are tacitly assuming that we have chosen u_i such that $V_p(u_1, u_2, \dots, u_{n-1}, 1) \neq 0$. This is always possible since $V_p \neq 0$. ■

Remark 10.5. Such power series expansions as in V above may not be defined, but we might still have a positive definite C^1 Lyapunov function. E.g.

$$V(x) = \begin{cases} x^2 & x \geq 0 \\ x^4 & x < 0 \end{cases}$$

△

Lemma 10.2 (*Invariant Sets*)

If $x = 0$ is an asymptotically stable equilibrium point, then its region of attraction $\mathcal{R}_A(0)$ is an open, invariant set. Moreover, $\partial\mathcal{R}_A(0)$ the boundary of $\mathcal{R}_A(0)$, is formed by trajectories. □

Proof of Lemma 10.2

Let $\phi_s^f(x)$ denote the solution to $\dot{y} = f(y)$, with initial condition $y(0) = x$, $s \in \mathbb{R}$. We wish to show that $\phi_s^f(x) \in \mathcal{R}_A(0)$ whenever $x \in \mathcal{R}_A(0), \forall s \in \mathbb{R}$.

By the semi-group property of solutions to ODEs,

$$\begin{aligned}
\phi_t^f(x(s)) &= \phi_t^f(\phi_s^f(x)) \\
&= \phi_{t+s}^f(x) \\
\lim_{t\to\infty} \phi_t^f(x(s)) &= \lim_{t\to\infty} \phi_{t+s}^f(x) \\
&\in \mathcal{R}_A(0) \qquad (\textit{by definition}).
\end{aligned}$$

Thus, $\phi_s^f(x) \in \mathcal{R}_A(0) \quad \forall s \in \mathbb{R}$*, whenever* $x \in \mathcal{R}_A(0)$*. (Invariance).*

To prove openness of $\mathcal{R}_A(0)$*, let* $p \in \mathcal{R}_A(0)$*. Let* $T > 0$ *be sufficiently large that*

$$\left\|\phi_T^f(p)\right\| < \frac{a}{2}$$

where $a > 0$ *is such that* $\{x : \|x\| < a\} \subset \mathcal{R}_A(0)$*. We can choose* b *small enough such that* $\forall q \in \{x : \|x - p\| < b\}$*, the solution* $\left\|\phi_T^f(p) - \phi_T^f(q)\right\| < \frac{a}{2}$*.*

Hence,

$$\begin{aligned}
\left\|\phi_T^f(q)\right\| &\leq \left\|\phi_T^f(p) - \phi_T^f(q)\right\| + \left\|\phi_T^f(p)\right\| \\
&< a. \\
&\implies \phi_T^f(q) \in \mathcal{R}_A(0) \\
&\implies \lim_{t\to\infty} \phi_t^f(q) = 0 \\
&\implies q \in \mathcal{R}_A(0).
\end{aligned}$$

For any open invariant set M*,* $x \in \partial M \implies \{x_n\} \subset M$ *such that* $\lim_{n\to\infty} x_n = x$*.*

Hence,

$$\begin{aligned}
&\{\phi_t^f(x_n) : t \in \mathbb{R}\} \subset M \\
&\lim_{t\to\infty} \phi_t^f(x_n) = \phi_t^f(x) \\
&\implies \phi_t^f(x) \quad \textit{is an accumulation point of } M \quad \forall t \in \mathbb{R}.
\end{aligned}$$

But $\phi_t^f(x) \notin M$*, since* $x \in \partial M$ *and* M *is open. Therefore,* $\phi_t^f(x) \in \partial M \quad \forall t \in \mathbb{R}$*. Thus,* ∂M *is made up of trajectories.* ■

Lecture 11

Stability Theory: Time-Varying Systems

The discussion of stability properties in time-varying systems is made complicated by the fact that dependence on initial conditions (specifically initial time) has an effect on how perturbations evolve. We limit ourselves to the study of stability of equilibria.

11.1. A Change of Variables

Suppose $\tau \mapsto \bar{y}(\tau)$ is a solution to the differential equation

$$\frac{dy}{d\tau} = g(\tau, y) \qquad \tau \geq a. \tag{11.1}$$

Consider the change of variables,

$$x(t) = y(t) - \bar{y}(\tau) \tag{11.2}$$

$$t = \tau - a \tag{11.3}$$

Then,

$$\begin{aligned}
\dot{x} \triangleq \frac{dx}{dt} &= \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} \\
&= \left(\frac{dy}{d\tau} - \frac{d\bar{y}}{d\tau} \right) \cdot 1 \\
&= g(\tau, y) - \frac{d\bar{y}}{d\tau} \\
&= g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \\
&\triangleq f(t, x)
\end{aligned}$$

The zero solution to this equation is given by

$$\begin{aligned} f(t,0) &= g(t+a,\bar{y}(t+a)) - \dot{\bar{y}}(t+a) \\ &= 0 \qquad t \geq a \qquad \text{(by equation (11.1))} \end{aligned}$$

Thus, examining the stability properties of the zero solution of $\dot{x} = f(t,x)$ is equivalent to examining the stability properties of the solution $\bar{y}$ of equation (11.1).

Remark 11.1. If g is not explicitly dependent on time and the solution $\bar{y}$ is nonconstant, this approach still leads to a necessarily non-autononmous transformed system $\dot{x} = f(t,x)$, due to the term $\dot{\bar{y}}(t+a)$. △

We are now ready for the basic definitions.

11.2. δ–ϵ Notions of Stability

Definition 11.1 (Stability). *The origin $x = 0$ is a stable equilibrium for the system $\dot{x} = f(t,x)$ if*

(i) $f(t,0) \equiv 0 \qquad \forall t \geq 0$

(ii) given $\epsilon > 0$ and any $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \implies \left\|\phi_t^f(t_0, x(t_0))\right\| < \epsilon \qquad \forall t \geq t_0$$

(Here $\phi_t^f(t_0,x)$ is the solution starting at t_0 at x).

△

The constant δ in general will depend on t_0. This means that if you wish to "trap" the solution in a ball of size ϵ, starting later might mean you are perhaps restricted to even smaller ball of perturbations about zero. To see this, consider the next example.

Example 11.1.

$$\dot{x} = (6t\sin t - 2t)x$$

This has the following solution passing through $x(t_0)$:

$$\begin{aligned} x(t) &= x(t_0)\exp\left(\int_{t_0}^{t}(6\sigma\sin\sigma - 2\sigma)d\sigma\right) \\ &= x(t_0)\exp\left(6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\right) \\ \implies |x(t)| &\leq |x(t_0)|c(t_0) \qquad t \geq t_0 \end{aligned}$$

Thus, for $\epsilon > 0$ choose $\delta = \frac{\epsilon}{c(t_0)}$. Then

$$\begin{aligned} |x(t_0)| < \delta &\implies |x(t)| < \epsilon \\ &\implies \text{Stability} \end{aligned}$$

<u>BUT</u>: $c(t_0) = \exp(-6 \sin t_0 + 6t_0 \cos t_0 + t_0^2 + K)$, where K is a constant, grows with t_0, so δ shrinks as t_0 increases!

We need a stronger notion of stability.

Definition 11.2 (Uniform stability)**.** *The equilibrium point 0 of $\dot{x} = f(t,x)$ is <u>uniformly stable</u> if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ independent of t_0 such that*

$$\|x(t_0)\| < \delta \implies \left\|\phi_t^f(t_0, x(t_0))\right\| < \epsilon \qquad \forall t \geq t_0.$$

△

The corresponding asymptotic definition is given next.

Definition 11.3 (Uniform asymptotic stability)**.** *The equilibrium point 0 of $\dot{x} = f(t,x)$ is <u>uniformly asymptotically stable</u> if*

(i) it is uniformly stable, and

(ii) there exists $c > 0$ independent of t_0 such that for $x(t)$ (the solution starting at $x(t_0)$), $x(t) \to 0$ as $t \to \infty$ uniformly in t_0, $\forall \|x(t_0)\| < c$.

△

11.3. Class $\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$ Notions of Stability

The devices of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ functions (associated with <u>$\mathcal{K}$amke</u>, an Austrian mathematician), provide another path to defining notions of stability for time-varying systems. We first provide background on these functions.

Definition 11.4 (Class $\mathcal{K}$)**.** *$\alpha : [0,a) \to [0,\infty)$ is <u>class $\mathcal{K}$</u> if*

(i) α is continuous,

(ii) $\alpha(0) = 0$, and

(iii) α is strictly increasing.

$\triangle$

Definition 11.5 (Class $\mathcal{K}_\infty$). $\alpha : [0, \infty) \to [0, \infty)$ *is* *class $\mathcal{K}_\infty$* *if*

(i) it is class $\mathcal{K}$, and

(ii) $\alpha(x) \to \infty$ as $x \to \infty$ (no leveling off).

$\triangle$

Definition 11.6 (Class $\mathcal{KL}$). $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is* *class $\mathcal{KL}$* *if*

(i) β is continuous jointly in the arguments

(ii) $\beta(\cdot, s)$ is of class $\mathcal{K}$ for each fixed s

(iii) $\beta(r, \cdot)$ is decreasing for each fixed r

(iv) $\beta(r, s) \to 0$ as $s \to \infty$.

$\triangle$

Example 11.2.
- $\alpha(r) = \arctan(r)$ is class $\mathcal{K}$ but not K_∞.
- $\alpha(r) = r^c$, $c > 0$ is class $\mathcal{K}_\infty$.
- $\beta(r, s) = r^c \exp(-s)$ is class $\mathcal{KL}$ for $c > 0$.

Properties 1. Let α_1, α_2 be class $\mathcal{K}$, let α_3, α_4 be class $\mathcal{K}_\infty$, and let $\beta(\cdot, \cdot)$ be class $\mathcal{KL}$. Then

(i) $\alpha_1^{-1} : [0, \alpha_1(a)) \to [0, \infty)$ is of class $\mathcal{K}$

(ii) α_3^{-1} is of class $\mathcal{K}_\infty$

(iii) $\alpha_1 \circ \alpha_2$ is of class $\mathcal{K}$

(iv) $\alpha_3 \circ \alpha_4$ is of class $\mathcal{K}_\infty$

(v) $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is of class $\mathcal{KL}$.

$\triangle$

Definition 11.7 (Uniform stability). *The equilibrium point* 0 *of the time-varying system* $\dot{x} = f(t,x)$ *is* uniformly stable *if there exists a class* $\mathcal{K}$ *function* $\alpha(\cdot)$ *and a positive constant* c*, independent of* t_0*, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \qquad t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c.$$

△

Definition 11.8 (Uniform asymptotic stability). *The equilibrium point* 0 *of the time-varying system* $\dot{x} = f(t,x)$ *is* uniformly asymptotically stable *if there exists a class* $\mathcal{KL}$ *function* $\beta(\cdot,\cdot)$ *and a positive constant* c*, independent of* t_0*, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c.$$

△

Remark 11.2. We can append the term global if the requirements of either definition 11.7 or 11.8 hold $\forall x(t_0) \in \mathbb{R}^n$. △

Remark 11.3. We can call the uniform asymptotic stability exponential stability if the requirements of definition 11.8 hold with

$$\beta(r,s) = kr\exp(-\gamma s), \qquad k, \gamma > 0.$$

△

Lemma 11.1 (*Equivalence of definitions*)

Definitions 11.2 and 11.7 are equivalent. Definitions 11.3 and 11.8 are equivalent. □

Proof of Lemma 11.1

Left as an exercise. ■

Exercise 11.1. Prove Lemma 11.1. (Hint: for the part of showing that 11.2 and 11.7 are equivalent, given $\epsilon > 0$ take $\delta = \alpha^{-1}(\epsilon)$).

11.4. Time-Varying Lyapunov Theory

Theorem 11.1 (*Time-Varying Lyapunov Theorem*)

Consider the system $\dot{x} = f(t,x)$*, satisfying* $f(t,0) \equiv 0 \ \forall t \geq 0$*, where* $f : [0,\infty) \times D \to \mathbb{R}^n$ *is piecewise continuous in* t *and locally Lipschitz in* x*. Let* $D = \{x \in \mathbb{R}^n : \|x\| < r\} = B_r(0)$*. Let* $V : [0,\infty) \times D \to \mathbb{R}$ *be a continuously* *differentiable function such that*

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|) \tag{11.4}$$

(positive definite) *(decrescent)*

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f \le -\alpha_3(\|x\|) \tag{11.5}$$

$\forall t \ge 0, \forall x \in D$, *for three class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ defined on $[0, r)$.*

Then $x = 0$ is uniformly asymptotically stable. □

Proof of Theorem 11.1

Let $\rho < r$. Define,

$$\Omega_{t,\rho} \triangleq \{x \in B_r(0) : V(t,x) \le \alpha_1(\rho)\}$$

Define $\tilde{\rho} \triangleq \alpha_2^{-1}(\alpha_1(\rho))$.
Now, $\|x\| \le \alpha_2^{-1}(\alpha_1(\rho)) \implies \alpha_2(\|x\|) \le \alpha_1(\rho)$. But, by the decresence property of V,

$$V(t,x) \le \alpha_2(\|x\|).$$

Thus,

$$\begin{aligned} \|x\| \le \alpha_2^{-1}(\alpha_1(\rho)) &\implies V(t,x) \le \alpha_1(\rho) \\ &\implies x \in \Omega_{t,\rho} \end{aligned}$$

We have shown,

$$B_{\tilde{\rho}}(0) \triangleq B_{\alpha_2^{-1}(\alpha_1(\rho))}(0) \subseteq \Omega_{t,\rho}$$

Further, $x \in \Omega_{t,\rho}$, i.e., $V(t,x) \le \alpha_1(\rho)$,

$$\begin{aligned} &\implies \alpha_1(\|x\|) \le \alpha_1(\rho) \qquad (\text{by the positive definiteness of } V) \\ &\implies \|x\| \le \rho. \end{aligned}$$

We have shown

$$\Omega_{t,\rho} \subseteq B_\rho$$

Taken together,

$$\implies B_{\tilde{\rho}} \subseteq \Omega_{t,\rho} \subseteq B_\rho \qquad \forall t \ge 0.$$

We have therefore verified the picture in Figure 11.1.

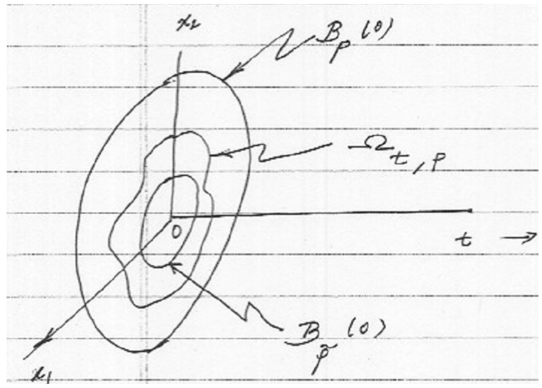


Figure 11.1: Illustration for set inclusions in UAS proof.

Since

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t, x) < 0$$

on $B_r(0) - \{0\}$, *by hypothesis, if for* $x_0 \in \Omega_{t,\rho}$, *the solution starting at* (t_0, x_0) *stays in* $\Omega_{t,\rho} \quad \forall t \geq t_0$.

Assume $x_0 \in B_{\tilde{\rho}}(0)$. *By the picture,* $x_0 \in \Omega_{t,\rho}$. *This implies,*

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V)) \quad (\textit{since } V(t,x) \leq \alpha_2(\|x\|) \textit{ by hypothesis}) \\ &\triangleq -\alpha(V). \end{aligned}$$

The function $\alpha(\cdot)$ *is class* $\mathcal{K}$ *on* $[0, \alpha_1(\rho))$.

Inspired by the above differential inequality, $\dot{V} \leq -\alpha(V)$, *we consider the differential equation*

$$\dot{y} = -\alpha(y).$$

(We will assume α *is locally Lipschitz. If not, there exists* $\tilde{\alpha}$ *locally Lipschitz, such that* $\alpha \geq \tilde{\alpha}$ *and we will use* $\tilde{\alpha}$ *instead of* α.*)*

One can prove (using a technical Lemma based on scalar differential inequalities—see supplemental material) that there is class $\mathcal{KL}$ *function* $\sigma(r, s)$ *defined on* $[0, \alpha_1(\rho)) \times [0, \infty) \to [0, \infty)$ *such that*

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0) \qquad \forall V(t_0, x(t_0)) \in [0, \alpha_1(\rho)).$$

Thus, for any solution starting at t_0 *, in* $B_{\tilde{\rho}} \subset \Omega_{t_0,\rho}$,

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1}(V(t, x(t))) \\ &\leq \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t - t_0)) \\ &\leq \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)) \\ &\triangleq \beta(\|x(t_0)\|, t - t_0), \end{aligned}$$

and β *is class* $\mathcal{KL}$. ■

Remark 11.4. In Theorem 11.1, $B_{\tilde{\rho}}(0)$ is an estimate of the domain of attraction. $\triangle$

Corollary 11.1

Let all hypotheses in Theorem 11.1 be global ($r \to \infty$), and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Then, 0 is globally uniformly asymptotically stable. $\square$

Proof of Corollary 11.1

$$\alpha_2^{-1} \circ \alpha_1 \in \mathcal{K}_\infty \implies \alpha_2^{-1}(\alpha_1(\rho)) \to \infty \text{ as } \rho \to \infty$$

This property is a substitute for radial unboundness. For any x_0, $\exists \rho > 0$ such that $\|x_0\| \leq \alpha_2^{-1}(\alpha_1(\rho))$. Then, the rest of the argument follows as in Theorem 11.1. $\blacksquare$

Corollary 11.2

If $\alpha_i(r) = k_i r^c$ with $k_i > 0$ and $c > 0$, then 0 is (uniformly) exponentially stable. $\square$

Proof of Corollary 11.2

Track the manipulation of the class $\mathcal{K}$ functions in the proof of Theorem 11.1.

$$\begin{aligned} \alpha(r) &= \alpha_3(\alpha_2^{-1}(r)) \\ &= k_3 \left(\left(\frac{r}{k_2} \right)^{1/c} \right)^c \\ &= \frac{k_3}{k_2} r \qquad \text{locally Lipschitz.} \end{aligned}$$

Then,

$$\begin{aligned} \implies \sigma(r,s) &= r \exp\left(-\frac{k_3}{k_2} s \right) \\ \implies \beta(r,s) &= \alpha_1^{-1}(\sigma(\alpha_2(r,s),s)) \\ &= \left(\frac{k_2}{k_1} r^c \exp\left(-\frac{k_3}{k_2} s \right) \right)^{1/c} \\ &= \left(\frac{k_2}{k_1} \right)^{1/c} r \exp\left(-\frac{k_3}{c k_2} s \right) \end{aligned}$$

Therefore, we take

$$k = \left(\frac{k_2}{k_1} \right)^{1/c} \qquad \text{and} \qquad r = \frac{k_3}{c k_2}.$$

$\blacksquare$

Lecture 12

Stability Theory: Time-Varying Systems (Linear Case)

For this lecture, we first specialize our study to linear time-varying (LTV) systems. We also study linear T-periodic systems. However, we will later return (next lecture) to nonlinear (time-varying) systems, making use of their linearization and the results we develop in the linear case.

12.1. Linear Time-Varying Systems

In the setting of linear time-varying systems, some of the ideas concerning uniform stability coalesce, as in the following theorem.

Theorem 12.1 (*LTV UAS/UES Theorem*)

Let $\dot{x} = A(t)x(t)$ be a linear system with piecewise continuous coefficient matrix $A(t)$. Then, the origin is uniformly asymptotically stable if and only if

$$\|\Phi(t, t_0)\| \leq k \exp\left(-\gamma(t - t_0)\right) \tag{12.1}$$

for some $k > 0$ and $\gamma > 0$. (That is, uniform asymptotic stability is equivalent to exponential stability in linear systems.) □

Proof of Theorem 12.1

Sufficiency: trivial.
Necessity: there exists $\beta(\cdot, \cdot)$ of class $\mathcal{KL}$ such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0, \quad \forall x(t_0) \in \mathbb{R}^n$$

$$\begin{aligned}\|\Phi(t,t_0)\| &\triangleq \max_{\|y\|=1} \|\Phi(t,t_0)y\| \\ &\leq \max_{\|y\|=1} \beta(\|y\|, t-t_0) \qquad (\textit{since } y \textit{ starting at } t_0 \textit{ is } \Phi(t,t_0)y) \\ &= \beta(1, t-t_0).\end{aligned}$$

Since $\beta(1,s) \to 0$ as $s \to \infty$, there exists $T > 0$ such that $\beta(1,t) \leq \frac{1}{e}$, $\forall t \geq T$. For every $t \geq t_0$, let N be the smallest positive integer such that $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each. Using the transition property of $\Phi(t,t_0)$, we can write

$$\Phi(t,t_0) = \Phi(t, t_0 + (N-1)T)\Phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \Phi(t_0 + T, t_0)$$

Then,

$$\begin{aligned}\|\Phi(t,t_0)\| &\leq \|\Phi(t, t_0 + (N-1)T)\| \cdot \prod_{k=1}^{N-1} \|\Phi(t_0 + KT, t_0 + (k-1)T)\| \\ &\leq \beta(1,0)\left(\frac{1}{e}\right)^{N-1} \\ &\leq e\beta(1,0)\exp\left(-\frac{t-t_0}{T}\right) \\ &= k\exp(-\gamma(t-t_0)),\end{aligned}$$

where $k \triangleq e\beta(1,0)$ and $\gamma \triangleq \frac{1}{T}$. ■

Remark 12.1. For time-varying linear systems, there are no simple tests based on eigenvalues to ascertain stability. One needs to use this theorem. However, in the case that $A(t)$ is periodic in t, the Floquet-Lyapunov Theorem does give a test for uniform asymptotic stability. We will discuss this in more detail later in the lecture. △

12.2. A Converse Theorem for LTV Systems

In this section, we discuss the existence of Lyapunov functions for systems that demonstrate asymptotic stability properties (a theorem in the class of so-called converse Lyapunov theorems)

Theorem 12.2 (*Converse Lyapunov Theorem for LTV Systems*)

Let $x = 0$ be an uniformly asymptotically stable equilibrium of $\dot{x} = A(t)x(t)$. Let $A(t)$ be continuous, $\|A(t)\|_2 \leq L$, $\forall t \geq 0$. Let $Q(t)$ be continuous, sym-

metric positive definite such that, for suitable constants c_3 and c_4

$$0 < c_3 I \leq Q(t) \leq c_4 I \qquad \forall t \geq 0.$$

then, there exists a unique, symmetric positive definite $P(t)$ satisfying

$$-\dot{P} = A^T P + PA + Q$$

and $P > 0$ is bounded above and below, such that for suitable constants c_1, c_2,

$$0 < c_1 I \leq P(t) \leq c_2 I \qquad \forall t \geq 0.$$

Hence, $V(t,x) = x^T P(t) x$ is a time-varying Lyapunov function for the given linear system, in the sense of the Time-Varying Lyapunov Theorem. □

Proof of Theorem 12.2

First recall that the notation

$$aI \leq M \leq bI$$

means

$$ay^T y \leq y^T M y \leq b y^T y \qquad \forall y \in \mathbb{R}^n$$

$$\iff \qquad a \leq \frac{y^T M y}{y^T y} \leq b$$

$$\iff \qquad a \leq \lambda_{\min}(M) \leq \lambda_{\max}(M) \leq b.$$

Now define

$$P(t) = \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau \qquad (\textit{where } \Phi \textit{ is the state transition matrix}).$$

It is easy to check that $P(t)$ is the only solution of

$$-\dot{P} = A^T P + PA + Q.$$

(Note: A, P, Q all depend on time t).

Let $\phi_\tau^A(t,x)$ denote the solution at time τ of the linear system starting at x at time t. Then, by linearity,

$$\phi_\tau^A(t,x) = \Phi(\tau,t)x.$$

Then,

$$\begin{aligned}
V(t,x) &= x^T P(t) x \\
&= x^T \left(\int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau \right) x \\
&= \int_t^\infty \phi_\tau^A(t,x)^T Q(\tau) \phi_\tau^A(t,x) d\tau \\
&\le \int_t^\infty c_4 \left\| \phi_\tau^A(t,x) \right\|_2^2 d\tau \\
&\le \int_t^\infty c_4 \left\| \Phi(\tau,t) \right\|_2^2 \cdot \|x\|_2^2 d\tau \\
&\le \int_t^\infty c_4 k^2 \exp(-2\gamma(\tau - t)) \|x\|_2^2 d\tau \\
&\le \frac{k^2 c_4}{2\gamma} \|x\|_2^2 \\
&\triangleq c_2 \|x\|_2^2
\end{aligned}$$

On the other hand, since $\|A(t)\|_2 \le L \quad \forall t \ge 0$*, by hypothesis, one can show that*

$$\frac{d}{dt} \left\| \phi_\tau^A(t,x) \right\|_2^2 \ge -2L \left\| \phi_\tau^A(t,x) \right\|_2^2.$$

This implies that

$$\left\| \phi_\tau^A(t,x) \right\|_2^2 \ge \|x\|_2^2 \exp(-2L(\tau - t)).$$

Hence,

$$\begin{aligned}
V(t,x) &= \int_t^\infty \phi_\tau^A(t,x)^T Q(\tau) \phi_\tau^A(t,x) d\tau \\
&\ge \int_t^\infty c_3 \left\| \phi_\tau^A(t,x) \right\|_2^2 d\tau \\
&\ge \int_t^\infty c_3 \exp(-2L(\tau - t)) \|x\|_2^2 d\tau \\
&= \frac{c_3}{2L} \|x\|_2^2 \\
&\triangleq c_1 \|x\|_2^2
\end{aligned}$$

Thus, we have

$$c_1 \|x\|_2^2 \le V(t,x) = x^T P(t) x \le c_2 \|x\|_2^2$$

with $c_1 = \frac{c_3}{2L}$ *and* $c_2 = \frac{k^2 c_4}{2\gamma}$.

Furthermore,

$$\begin{aligned}\dot{V} &= \frac{\partial V(t,x)}{\partial t} + \frac{\partial V}{\partial x} \cdot A(t)x \\ &= x^T(\dot{P} + A^T P + PA)x \\ &= -x^T Q(t)x \\ &\leq -c_3 \|x\|_2^2\end{aligned}$$

Thus, $V(t,x) = x^T P(t)x$ is a time-dependent Lyapunov function satisfying

$$\begin{aligned}\alpha_1(\|x\|) \leq V(t,x) &\leq \alpha_2(\|x\|) \\ \dot{V}(t,x) &\leq -\alpha_3(\|x\|)\end{aligned}$$

where the class $\mathcal{K}$ functions α_i are given by

$$\alpha_i(y) = c_i \cdot y^2,$$

with the c_i constants defined above. ■

Remark 12.2. The formula

$$V(t,x) = \int_t^\infty \phi_\tau^A(t,x)^T Q(\tau)\phi_\tau^A(t,x)d\tau$$

suggests a possible path to converse Lyapunov theorems for nonlinear systems—let $\phi_\tau^f(t,x)$ be the solution starting at x at t for the nonlinear system. △

12.3. T-Periodic LTV systems

Theorem 12.3 (*Periodic Linear Systems - Floquet*)

Consider $\dot{x} = A(t)x(t)$, with $A(t)$ piecewise continuous, $x(t_0) = x_0$, and $A(t+T) = A(t), \forall t$. Let Φ denote the state transition matrix.

Then

(i) $\Phi(t+T, t_0+T) = \Phi(t,t_0)$

(ii) There is a constant matrix R and a T-periodic nonsingular matrix function $P(t)$ such that

$$\Phi(t,t_0) = P^{-1}(t)\exp(R(t-t_0))P(t_0).$$

(iii) 0 is uniformly (asymptotically) stable for the given system if and only if it is uniformly (asymptotically) stable for the system

$$\dot{z} = Rz.$$

□

Proof of Theorem 12.3

(i) Recall the Peano-Baker series, and express it for a time period T,

$$\begin{aligned}
\Phi(t+T, t_0+T) &= I + \int_{t_0+T}^{t+T} A(\sigma_1)d\sigma_1 + \int_{t_0+T}^{t+T}\int_{t_0+T}^{\sigma_1} A(\sigma_1)A(\sigma_2)d\sigma_2 d\sigma_1 + \cdots \\
&= I + \int_{t_0+T}^{t+T} A(\sigma_1+T)d\sigma_1 \\
&\quad + \int_{t_0+T}^{t+T}\int_{t_0+T}^{\sigma_1} A(\sigma_1+T)A(\sigma_2+T)d\sigma_2 d\sigma_1 + \cdots \\
&\qquad\qquad (\text{by } T\text{-periodicity of } A) \\
&= I + \int_{t_0}^{t} A(\hat{\sigma}_1)d\hat{\sigma}_1 + \int_{t_0}^{t}\int_{t_0}^{\hat{\sigma}_1} A(\hat{\sigma}_1)A(\hat{\sigma}_2)d\hat{\sigma}_2 d\hat{\sigma}_1 + \cdots \\
&\qquad\qquad (\text{by change of variables}) \\
&= \Phi(t, t_0)
\end{aligned}$$

(ii) $\Phi(T,0)$ is nonsingular and hence has a (possibly complex matrix) logarithm RT, i.e.,

$$\Phi(T,0) = \exp(RT) \tag{12.2}$$

Let

$$\begin{aligned}
P^{-1}(t) &\triangleq \Phi(t,0)\exp(-RT) \qquad \text{(Note: } P(0) = I\text{)} \\
P^{-1}(t+T) &= \Phi(t+T,0)\exp(-R(t+T)) \\
&= \Phi(t+T,T)\Phi(T,0)\exp(-RT)\exp(-Rt) \\
&= \Phi(t,0)\exp(-Rt) \\
&= P^{-1}(t)
\end{aligned}$$

Hence, P^{-1} is T-periodic.
Thus,

$$\begin{aligned}
\Phi(t,t_0) &= \Phi(t,0)\Phi(0,t_0) \\
&= \Phi(t,0)(\Phi(t_0,0))^{-1} \\
&= P^{-1}(t)\exp(Rt)(P^{-1}(t_0)\exp(Rt_0))^{-1} \\
&= P^{-1}(t)\exp(Rt)\exp(-Rt_0)P(t_0) \\
&= P^{-1}(t)\exp(R(t-t_0))P(t_0)
\end{aligned}$$

(iii) Let

$$
\begin{aligned}
z(t) &= P(t)x(t) \\
\dot{z} &= \dot{P}x + P\dot{x} \\
&= \frac{d}{dt}((P^{-1})^{-1})x + P\dot{x} \\
&= -P\frac{d}{dt}(P^{-1})Px + P\dot{x} \\
&\qquad \left(\text{matrix-scalar differentiation: } \frac{\partial U^{-1}}{\partial t} = -U^{-1}\frac{\partial U}{\partial t}U^{-1}\right) \\
&= -P\frac{d}{dt}\left(\Phi(t,0)\exp(-Rt)\right)Px + P\dot{x} \\
&= -\left(P\frac{d}{dt}(\Phi(t,0))\exp(-Rt)P + P\Phi(t,0)\frac{d}{dt}(\exp(-Rt))P\right)x + PAx \\
&= -\left(PA\Phi(t,0)\exp(-Rt)P + P\Phi(t,0)\exp(-Rt)(-R)P\right)x + PAx \\
&= -\left(PAP^{-1}P + PP^{-1}(-R)P\right)x + PAx \\
&= -PAx + RPx + PAx \\
&= Rz
\end{aligned}
$$

P has piecewise continuous derivatives on $(-\infty,\infty)$; $\dot{P} = RP - PA$; P and $\dot{P}$ are bounded on $(-\infty,\infty)$ because they are piecewise continuous and T-periodic; Furthermore, by these properties, there exist $m_1, m_2 > 0$ such that

$$0 < m_1 \le |\det P(t)| \le m_2.$$

Hence,

$$\|z(t)\| < c_1\|x(t)\|$$

and

$$\|x(t)\| < c_2\|z(t)\|$$

where

$$
\begin{aligned}
c_1 &= \max_{[t,t+T]}\|P(t)\| \\
c_2 &= \max_{[t,t+T]}\left\|P^{-1}(t)\right\|
\end{aligned}
$$

From these inequalities, it follows that all the stability properties of z carry over to those of x and vice versa. ∎

Corollary 12.1 (*Converse Lyapunov Theorem for T-Periodic LTV Systems*)

Consider $\dot{x} = A(t)x(t)$, with $A(t+T) = A(t)$ be piecewise continuous and T-periodic. Let $x = 0$ be uniformly, asymptotically stable equilibrium of $\dot{x}(t) = A(t)x(t)$. Then there is a T-periodic Lyapunov function

$$V = V(t,x) = V(t+T,x) = x^T P(t)x$$

satisfying

$$-\dot{P} = A^T P + PA + Q$$

for each T-periodic Q and

$$0 < c_3 I < Q(t) \leq c_4 I.$$

□

Proof of Corollary 12.1

Essentially the same construction as in Theorem 12.2. ■

Corollary 12.2 (*Floquet-Lyapunov*)

Consider $\dot{x} = A(t)x(t)$, with $A(t+T) = A(t)$ be piecewise continuous and T-periodic. If all eigenvalues of $\Phi(T,0)$ are inside the open unit disk $\{z : |z| < 1\}$ in the complex plane, then the system is uniformly asymptotically stable. □

Example 12.1. Consider the system $\dot{x} = A(t)x(t)$ with following 2π-periodic $A(t)$ matrix,

$$A(t) = \begin{bmatrix} -1+\cos t & 0 \\ 0 & -2+\cos t \end{bmatrix}.$$

Recall from linear systems theory for

$$\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0) \quad \text{with} \quad \Phi(0,0) = I,$$

that if $A(t)$ commutes with its integral,

$$A(t)\int_\tau^t A(\sigma)d\sigma = \left[\int_\tau^t A(\sigma)d\sigma\right] A(t)$$

then the transition matrix is given by

$$\Phi(t,0) = \exp\left(\int_0^t A(\sigma)d\sigma\right).$$

Thus, for this problem we have

$$\begin{aligned}\Phi(2\pi, 0) &= \begin{bmatrix} \exp\left(\int_0^{2\pi}(-1+\cos t)dt\right) & 0 \\ 0 & \exp\left(\int_0^{2\pi}(-2+\cos t)dt\right) \end{bmatrix} \\ &= \begin{bmatrix} \exp(-2\pi) & 0 \\ 0 & \exp(-4\pi) \end{bmatrix}.\end{aligned}$$

Both eigenvalues are in the open unit disk, which implies that zero is uniformly asymptotically stable by the Floquet-Lyapunov Theorem.

Alternately, since

$$\Phi(T, 0) = \exp(RT)$$

in the proof of Theorem 12.3, we see that for this problem

$$R = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

which has all eigenvalues in the open left-half plane. By Theorem 12.3, $\dot{z} = Rz$ is uniformly asymptotically stable and so is $\dot{x}(t) = A(t)x(t)$.

Lecture 13

Stability Theory: Assessing via Linearization

Here we wish to state and prove a theorem on assessing stability of nonlinear systems via linearization. We need some background first.

13.1. Mathematical Background

The fundamental theorem of integral calculus was already stated in a previous lecture note, however it is so fundamental it is repeated here for reference. Proof is omitted.

Theorem 13.1 (*Fundamental Theorem of Integral Calculus*)

Let X and Y be two finite dimensional vector spaces. Let $U \overset{open}{\subset} X$ and let $f : U \to Y$ be C^1.

If $x + ty \in U, \forall t \in [0, 1]$ (e.g. if $U = B_r(x)$),
then

$$f(x + y) = f(x) + \int_0^1 Df(x + ty)y dt \tag{13.1}$$

(Note: $Df(z)h = \frac{d}{ds}(z + sh)\Big|_{s=0}$ is the Fréchet derivative.) □

Proof of Theorem 13.1

See previous lecture notes. ■

Remark 13.1. The theorem is true as stated when X, Y are general Banach spaces. However, we have to have a suitable theory of the integral. In finite dimensions, we are content with the Riemann integral. △

Taking equation (13.1), we can write

$$
\begin{aligned}
f(x) &= f(0) + \int_0^1 Df(tx)x dt \\
&= f(0) + M(x)x,
\end{aligned}
$$

where

$$M(x) \triangleq \int_0^1 Df(tx)dt$$

an x-dependent, matrix-valued function.

Consider a C^1 vector field $f(x)$ with $f(0) = 0$. Let

$$A \triangleq \left(\frac{\partial f}{\partial x}\right)\Big|_0 = (Df)(0). \tag{13.2}$$

Now let us rewrite f as

$$
\begin{aligned}
f(x) &= Ax + (f(x) - Ax) \\
&= Ax + g(x),
\end{aligned}
$$

where $g(x) \triangleq f(x) - Ax$. In this way, $g(x)$ represents the deviation of the f field from its linearization at x. We note that $g(\cdot)$ is C^1, since f is C^1. Applying the fundamental theorem of integral calculus, one can write,

$$g(x) = g(0) + N(x)x$$

where

$$
\begin{aligned}
g(0) &= f(0) - A(0) = 0, \text{and} \\
N(x) &= \int_0^1 Dg(tx)dt \\
&= \int_0^1 (Df(tx) - A)dt \\
\lim_{x\to 0} N(x) &= \int_0^1 (\lim_{x\to 0} Df(tx) - A)dt \\
&= \int_0^1 (Df(0) - A)dt \\
&= \int_0^1 (A - A)dt \\
&= 0
\end{aligned}
$$

Thus $\frac{\|g(x)\|}{\|x\|} \leq \|N(x)\| \to 0$ as $\|x\| \to 0$, in any norm.

Then, for any $\gamma > 0$ (arbitrarily small), there exists an $r > 0$ such that

$$\|g(x)\|_2 < \gamma \|x\|_2 \qquad \forall \|x\|_2 < r \tag{13.3}$$

This key property will be used below. We will also use the notation above (for $f(x), g(x), N(x)$, etc.) throughout the lecture.

Remark 13.2. In the non-autonomous case, with $f = f(t,x)$ and $f(t,0) \equiv 0$, then $g(t,x) = f(t,x) - A(t)x$ where

$$A(t) \triangleq \left(\frac{\partial f(t,x)}{\partial x}\right)\bigg|_{x=0}$$

has the property that

$$\frac{\|g(t,x)\|_2}{\|x\|_2} \to 0 \quad \text{as} \quad \|x\|_2 \to 0$$

for each $t \geq 0$. However, this property does NOT hold uniformly in general. That is, one cannot take for granted the following condition:

$$\lim_{\|x\|_2 \to 0} \left(\sup_{t \geq 0} \frac{\|g(t,x)\|_2}{\|x\|_2} \right) = 0.$$

This is known as the uniform order condition. Such a uniform hypothesis is needed for a linearization-based stability theorem for non-autonomous nonlinear systems. See Sastry pages 214-215 for further discussion.

Exercise 13.1. Evaluate the system $\dot{x} = f(t,x) = -x + tx^2$ with respect to the uniform order condition.

$\triangle$

Theorem 13.2 (*Indirect Method of Lyapunov - Time-Invariant Case*)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Assume f is C^1 on a neighborhood $B_\rho(0)$ of 0. Let $A = \frac{\partial f}{\partial x}\Big|_0$. If the spec(A) $\subseteq \mathbb{C}^-$, the open left-half plane, then the origin is an asymptotically stable equilibrium point of the nonlinear system. $\square$

Proof of Theorem 13.2

Let $Q = Q^T > 0$. Then there exists a unique $P > 0$ such that

$$A^T P + PA = -Q$$

Since A is Hurwitz,

$$P = \int_0^\infty \exp(A^T\sigma)Q\exp(A\sigma)d\sigma$$

is a convergent integral that is positive definite and solves this equation.

Let $V(x) = x^T P x$ and compute the derivative along trajectories of $\dot{x} = f(x)$,

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + g(x))^T P x + x^T P(Ax + g(x)) \\ &= x^t(A^T P + PA)x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x)\end{aligned}$$

But

$$\begin{aligned}x^T P g(x) &\leq \|x\|_2 \cdot \|Pg(x)\|_2 \qquad \text{(Cauchy-Schwarz)} \\ &\leq \|x\|_2 \cdot \|P\|_2 \cdot \|g(x)\|_2 \\ &< \gamma \|x\|_2 \cdot \|P\|_2 \cdot \|x\|_2 \qquad (\text{if } \|x\|_2 < r < \rho).\end{aligned} \tag{13.4}$$

On the other hand (by Rayleigh),

$$0 < \lambda_{\min}(Q)\|x\|_2^2 < x^T Q x < \lambda_{\max}(Q)\|x\|_2^2.$$

This implies

$$-x^T Q x < \lambda_{\min}(Q)\|x\|_2^2. \tag{13.5}$$

Taken together, inequalities (13.4) *and* (13.5) *imply that*

$$\begin{aligned}\dot{V} &< \lambda_{\min}(Q)\|x\|_2^2 + 2\gamma\|P\|_2 \cdot \|x\|_2^2 \\ &= (-\lambda_{\min}(Q) + 2\gamma\|P\|_2)\|x\|_2^2.\end{aligned}$$

Picking Q, determines $\lambda_{\min}(Q)$ and $\|P\|_2$. We can pick r sufficiently small so that γ is sufficiently small, yielding

$$-\lambda_{\min}(Q) + 2\gamma\|P\|_2 < 0.$$

By Lyapunov, we have asymptotic stability. (Note: of course a smaller r means the estimate of the domain of attraction $B_r(0)$ is smaller.) ∎

Theorem 13.3 (*Lyapunov's Indirect Instability Theorem*)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Assume f is C^1 and a neighborhood $B_\rho(0)$ of 0. Let $A = \left(\frac{\partial f}{\partial x}\right)\Big|_0$.

If spec$(A) \subseteq \mathbb{C}^+$, the open right-half plane, then the origin is an unstable equilibrium point of the nonlinear system. □

Proof of Theorem 13.3

By hypothesis, spec$(-A) \subseteq \mathbb{C}^-$. So for $Q = Q^T > 0$ there exists a unique $P > 0$ such that

$$(-A)^T P + P(-A) = -Q. \tag{13.6}$$

Along trajectories of $\dot{x} = f(x) = Ax + g(x)$, the derivative of $V(x) = x^T Px$ satisfies

$$\begin{aligned}
\dot{V} &= \dot{x}^T Px + x^T P\dot{x} \\
&= x^T(A^T P + PA)x + 2x^T Pg(x) \\
&= x^T Qx + 2x^T Pg(x) \\
&\geq \lambda_{\min}(Q) \left\| x \right\|_2^2 - 2|x \cdot Pg(x)| \\
&\geq \lambda_{\min}(Q) \left\| x \right\|_2^2 - 2 \left\| x \right\|_2 \cdot \left\| P \right\|_2 \cdot \left\| g(x) \right\|_2 \\
&> \lambda_{\min}(Q) \left\| x \right\|_2^2 - 2\gamma \left\| x \right\|_2^2 \cdot \left\| P \right\|_2,
\end{aligned}$$

(for $\left\| x \right\|_2 < r < \rho$, r sufficiently small).

Pick Q. This determines P and $\left\| P \right\|_2$. Pick r sufficiently small so that γ is sufficiently small, yielding

$$\begin{aligned}
\dot{V} &> (\lambda_{\min}(Q) - 2\gamma \left\| P \right\|_2) \left\| x \right\|_2^2 \\
&> 0 \qquad \forall x \in B_r(0) - \{0\}.
\end{aligned}$$

By Lyapunov's Instability I Theorem, it follows that 0 is unstable for the nonlinear system. ■

The hypotheses of Theorem 13.3 are rather strong. We can do better.

Theorem 13.4

In the statement of Theorem 13.3, assume that at least one of the eigenvalues of A is in $\mathbb{C}^+$. Then 0 is unstable. □

Proof of Theorem 13.4

In general A has a splitting of spectrum

$$\text{spec}(A) = \sigma_- \cup \sigma_0 \cup \sigma_+$$

where $\sigma_- \subseteq \mathbb{C}^-$, $\sigma_0 \subseteq j\omega$ axis, and $\sigma_+ \subseteq \mathbb{C}^+$. We have assumed that $\sigma_+ \neq \emptyset$.

Then there exists $\epsilon > 0$ such that

$$\text{spec}\left(A - \frac{\epsilon}{2} I\right) = \sigma_-^\epsilon \cup \sigma_+^\epsilon,$$

where $\sigma_+^\epsilon \neq \emptyset$. (We got rid of the pure imaginary eigenvalues by a right shift of the imaginary axis.)

Let $A^\epsilon \triangleq A - \frac{\epsilon}{2}I$.

There is a nonsingular, real matrix T (recall the real Jordan form) such that

$$\begin{aligned}
TA^\epsilon T^{-1} &= TAT^{-1} - \frac{\epsilon}{2}I \\
&= \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] - \frac{\epsilon}{2}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\
&= \left[\begin{array}{c|c} A_1 - \frac{\epsilon}{2}I & 0 \\ \hline 0 & A_2 - \frac{\epsilon}{2}I \end{array}\right]
\end{aligned}$$

where spec$(A_1 - \frac{\epsilon}{2}I) \subseteq \mathbb{C}^-$ and spec$(A_2 - \frac{\epsilon}{2}I) \subseteq \mathbb{C}^+$. Let $Q_i = Q_i^T > 0$ and let $P_i = P_i^T > 0$ be the unique matrices satisfying

$$\begin{aligned}
\left(A_1 - \frac{\epsilon}{2}I\right)^T P_1 + P_1\left(A_1 - \frac{\epsilon}{2}I\right) &= -Q_1 \\
-\left(A_2 - \frac{\epsilon}{2}\right)^T P_2 - P_2\left(A_2 - \frac{\epsilon}{2}\right) &= -Q_2
\end{aligned}$$

(We have used the fact that spec$(-(A_2 - \frac{\epsilon}{2})) \subseteq \mathbb{C}^-$).

Consider $z = Tx$. Then

$$\dot{z} = T(Ax + g(x))$$

where $g(x) \triangleq f(x) - Ax$. This implies

$$\begin{aligned}
\dot{z} &= TAT^{-1}z + Tg(T^{-1}z) \\
&= \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix}
\end{aligned}$$

By hypothesis and the definition of $g(\cdot)$, it follows that $h(0) = 0$. Given $\gamma > 0$, there exists $r > 0$ such that

$$\|h(z)\|_2 < \gamma \|z\|_2 \qquad \forall \|z\|_2 < r.$$

Define

$$V(z) = -z_1^T P_1 z_1 + z_2^T P_2 z_2.$$

Then

$$\begin{aligned}
\dot{V}(z) &= -\dot{z}_1^T P_1 z_1 - z_1^T P_1 \dot{z}_1 + \dot{z}_2^T P_2 z_2 + z_2^T P_2 \dot{z}_2 \\
&= -z_1^T (A_1^T P_1 + P_1 A_1) z_1 - 2 z_1^T P_1 h_1(z) + z_2^T (A_2^T P_2 + P_2 A_2) z_2 \\
&\qquad + 2 z_2^T P_2 h_2(z) \\
&= -z_1^T \Big(\Big(A_1 - \frac{\epsilon}{2} I\Big)^T P_1 + P_1 \Big(A_1 - \frac{\epsilon}{2} I\Big)\Big) z_1 - \epsilon z_1^T P_1 z_1 - 2 z_1^T P_1 h_1(z) \\
&\qquad + z_2^T \Big(\Big(A_2 - \frac{\epsilon}{2} I\Big)^T P_2 + P_2 \Big(A_2 - \frac{\epsilon}{2} I\Big)\Big) z_2 + \epsilon z_2^T P_2 z_2 + 2 z_2^T P_2 h_2(z) \\
&= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + \epsilon V(z) - 2 z_1^T P_1 h_1 + 2 z_2^T P_2 h_2 \\
&= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + \epsilon V(z) - 2 z^T \begin{bmatrix} P_1 h_1 \\ -P_2 h_2 \end{bmatrix} \\
&\geq \lambda_{\min}(Q_1) \|z_1\|_2^2 + \lambda_{\min}(Q_2) \|z_2\|_2^2 + \epsilon V(z) \\
&\qquad - 2 \|z\|_2 \cdot \max(\|P_1\|_2, \|P_2\|_2) \cdot \|h(z)\|_2^2 \\
&\geq \left(\alpha - 2\sqrt{2}\gamma\beta\right) \|z\|_2^2 + \epsilon V(z),
\end{aligned}$$

where $\alpha = \min(\lambda(Q_1), \lambda_2(Q_2))$ *and* $\beta = \max(\|P_1\|_2, \|P_2\|_2)$, $\forall \|z\|_2 < r$ *and* $\gamma > 0$.

Let $U = \{z \in B_r(0) : V(z) > 0\}$. *Then,* $\dot{V} > 0$ *on* U. *In fact, there is a quadratic class* $\mathcal{K}$ *function bounding* $\dot{V}$ *below– provided* $\gamma < \frac{\alpha}{2\sqrt{2}\beta}$.

By Chetaev's Instability Theorem, 0 *is unstable.* ■

Remark 13.3. The cases where there are no eigenvalues on the open right half plane, but there are eigenvalues on the imaginary axis are called critical cases. For these cases, one cannot say anything about stability via linearization. △

Lecture 14

Feedback Stabilization and Feedback Linearization

14.1. Feedback Stabilization

Consider the system

$$\dot{x} = f(x, u) \tag{14.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Suppose $f(0,0) = 0$ and f is C^1. Let $A = \left(\frac{\partial f}{\partial x}\right)\Big|_{(0,0)}$ and let $B = \left(\frac{\partial f}{\partial u}\right)\Big|_{(0,0)}$.

Hypothesis: Let K be such that spec$(A + BK) \subseteq \mathbb{C}^-$. (Typically, we choose K such that this is true.)

Consider the closed loop system

$$\begin{aligned} \dot{x} &= f(x, u) \\ u &= Kx. \end{aligned}$$

Thus $\dot{x} = \tilde{f}(x) = f(x, Kx)$. Clearly, we have $\tilde{f}(0) = 0$. The linearization of the closed loop system, at the origin is $\dot{z} = \tilde{A}z$, where

$$\tilde{A} = \left(\frac{\partial \tilde{f}}{\partial x}\right)\Big|_0$$

But

$$\left(\frac{\partial \tilde{f}}{\partial x}\right)\Big|_0 = \left(\frac{\partial f(x, Kx)}{\partial x}\right)\Big|_0$$
$$= D_1 f\Big|_{x=0} + D_2 f \cdot K\Big|_{x=0}$$
$$= (A + BK).$$

By our hypothesis and the indirect method of Lyapunov, the origin is an asymptotically stable equilibrium of a closed loop system.

Remark 14.1. A sufficient condition for our hypothesis to hold is that the pair $[A, B]$ be controllable (recall: the eignevalue/pole placement theorem). △

We see that $u = Kx$, a linear feedback law, can stabilizing. The region of attraction may be estimated by

(i) solving for P in $(P + BK)^T P + P(A + BK) = -Q$ with $Q = Q^T > 0$,

(ii) letting $g(x) = f(x, Kx) - (A + BK)x$ and observing that $\|g(x)\|_2 \leq \gamma \|x\|_2$ for all $\|x\|_2 < r$ and γ can be made arbitrarily small by choosing r small enough.

Then, if $(-\lambda_{\min}(Q) + 2\gamma \|P\|_2) < 0$, $B_r(0)$ is an estimate of the region of attraction centered at the origin. (This inequality is from an argument used in a previous lecture in the proof of Lyapunov's indirect method).

14.2. Feedback Linearization

The region of attraction $B_r(0)$ derived using linear feedback may be too small for practical purposes. One approach to overcome this problem is to use feedback and changes of coordinates in input space and state space to exactly linearize a nonlinear control system in a (sufficiently large) neighborhood of any equilibrium point. Even if linearization does not have asymptotic stability, controllability can ensure the existence of an additional feedback to stablize the system/equilibrium.

Definition 14.1. *Let*

$$\dot{y} = f(y) + G(y)u$$

where $f(0) = 0$, $G(y) = [g_1(y), \ldots, g_m(y)]$ *and* $g_i(0) = 0$. *We say that this system is exact state feedback linearizable if there exists*

$$T : U \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^n$$

$0 \in U$, (T is C^∞, and T^{-1} exists and is C^∞), and functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that under the change of coordinates given by T,

$$x = T(y)$$

satisfies

$$\begin{aligned}\dot{x} &= Ax + B\beta^{-1}[u - \alpha(x)] \\ &\triangleq Ax + Bv\end{aligned}$$

where $v \triangleq \beta^{-1}(x)[u - \alpha(x)]$ and $[A, B]$ is controllable. △

We thus seek $\beta(\cdot)$ such that $\beta(T(y))$ is an invertible $m \times m$ matrix at every y.

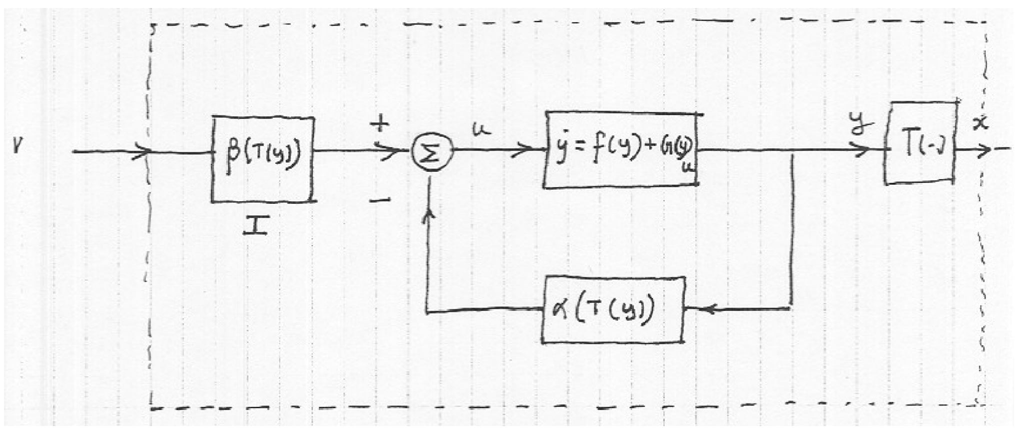


Figure 14.1: Feedback linearization diagram.

In Figure 14.1, the system outside the dotted line is linear.

By the chain rule,

$$\begin{aligned}\dot{x} &= \frac{\partial T}{\partial y}\dot{y} \\ &= \frac{\partial T}{\partial y}(f(y) + G(y)u) \\ &= Ax + B\beta^{-1}(x)[u - \alpha(x)] \\ &= AT(y) + B\beta^{-1}(T(y))[u - \alpha(T(y))] \qquad \forall y \in U \\ &= AT(y) + B\beta^{-1}(y)[u - \alpha(y)] \qquad \forall y \in U \\ &= Ax + Bv\end{aligned}$$

Set $u \equiv 0$. This implies,

$$\frac{\partial T}{\partial y}f(y) = AT(y) - B\beta^{-1}(y)\alpha(y). \qquad (14.2)$$

Equating terms that multiply the input, we also have,

$$\frac{\partial T}{\partial y} G(y) = B\beta^{-1}(y) \tag{14.3}$$

In equations 14.2 and 14.3, we have derived constraints that transformation T must satisfy.

For the specific case of a single input ($m = 1$), we can further specialize these constraints by assuming forms for the A and B matrices into which we would like to transform our nonlinear system. Consider the canonical form $A = A_c$; $B = B_c$; with $g \triangleq G$.

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This linear system corresponds to a chain of integrators with direct control over the $\dot{x}_n$ term with a single input. Then the conditions on T above in equations 14.2 and 14.3 take the form (assuming $\beta \neq 0$),

Equation #	Equation	Equation #	Equation
$(1a)$	$\frac{\partial T_1}{\partial y} f(y) = T_2(y)$	$(1b)$	$\frac{\partial T_1}{\partial y} g(y) = 0$
$(2a)$	$\frac{\partial T_2}{\partial y} f(y) = T_3(y)$	$(2b)$	$\frac{\partial T_2}{\partial y} g(y) = 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$((n-1)a)$	$\frac{\partial T_{n-1}}{\partial y} f(y) = T_n(y)$	$((n-1)b)$	$\frac{\partial T_{n-1}}{\partial y} g(y) = 0$
(na)	$\frac{\partial T_n}{\partial y} f(y) = -\frac{\alpha(y)}{\beta(y)}$	(nb)	$\frac{\partial T_n}{\partial y} g(y) = \frac{1}{\beta(y)}$

Define the operation (Lie Derivative):

$$L_f h = \frac{\partial h}{\partial x} \cdot f \tag{14.4}$$

where h is a scalar function and f is a vector field. We also define the convenient notation

$$L_f^0 \triangleq h \tag{14.5}$$

and

$$L_f^{k+1} h = L_f(L_f^k h). \tag{14.6}$$

With these definitions, from our constraints on T, we have the relations,

$$T_k = L_f^{k-1} T_1 \qquad k = 2, 3, \ldots, n. \tag{14.7}$$

and we also have

$$L_g L_f^k T_1 = 0 \qquad k = 0, 1, 2, \ldots, (n-2). \tag{14.8}$$

Equations 14.8 are partial differential equations for T_1, and if we can solve them for T_1, then by using the recursions above, we can define T_k for $k = 2, \ldots, n$ and

$$\beta(y) = (L_g T_n)^{-1} \tag{14.9}$$

if $\beta \neq 0$. Further, we also have

$$\alpha(y) = -\frac{L_f T_n}{L_g T_n}. \tag{14.10}$$

What about the solvability of equations 14.8 for T_1?

Define

$$\mathrm{ad}_f g \triangleq \left(\frac{\partial g}{\partial x}\right) f - \left(\frac{\partial f}{\partial x}\right) g \tag{14.11}$$

and also

$$\mathrm{ad}_f^0 g \triangleq g \tag{14.12}$$

$$\mathrm{ad}_f^{k+1} g \triangleq \mathrm{ad}_f(\mathrm{ad}_f^k g). \tag{14.13}$$

Theorem 14.1

There exists (locally in a suitable neighborhood of 0) a function T_1 such that

$$L_g L_f^k T_1 = 0 \qquad k = 0, 1, 2, \ldots, (n-2)$$

if and only if

(i) $\{g, \mathrm{ad}_f g, \mathrm{ad}_f^2 g, \ldots, \mathrm{ad}_f^{n-1} g\}$ *is a set of linearly independent vector fields.*

(ii) $\{g, \mathrm{ad}_f g, \mathrm{ad}_f^2 g, \ldots, \mathrm{ad}_f^{n-2} g\}$ *is a set of vector fields satisfying the involutive property, which states that for $p(x), q(x) \in$ this set,*

$$\mathrm{ad}_p q = \frac{\partial q}{\partial x} p(x) - \frac{\partial p}{\partial x} q(x)$$

also belongs to this set.

□

Remark 14.2. This existence result is a consequence of Frobenius' Theorem in differential geometry. △

Example 14.1.

$$\dot{y} = f(y) + g(y)u$$

$$f(y) = \begin{bmatrix} y_2 \\ -a\sin(y_1) - b(y_1 - y_3) \\ y_4 \\ c(y_1 - y_3) \end{bmatrix} \qquad g(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} \qquad \text{for } a, b, c, d > 0.$$

$$\begin{aligned}
L_g T_1 = 0 &\iff \frac{\partial T_1}{\partial y_4} = 0 \qquad \implies \qquad \underline{T_1 \text{ independent of } y_4}. \\
T_2 &= L_f T_1 \\
&= \frac{\partial T_1}{\partial y_1} y_2 + \frac{\partial T_1}{\partial y_2}(-a\sin y_1 - b(y_1 - y_3)) + \frac{\partial T_1}{\partial y_3} y_4. \\
L_g T_2 = 0 &\iff \frac{\partial T_2}{\partial y_4} \iff \frac{\partial T_1}{\partial y_3} = 0 \implies \underline{T_1 \text{ independent of } y_3}. \\
T_3 &= L_f T_2 \\
&= \frac{\partial T_2}{\partial y_1} y_2 + \frac{\partial T_2}{\partial y_2}(-a\sin y_1 - b(y_1 - y_3)) + \frac{\partial T_2}{\partial y_3} y_4
\end{aligned}$$

$$L_g T_3 = 0 \implies \frac{\partial T_3}{\partial y_4} = 0 \implies \frac{\partial T_2}{\partial y_3} = 0 \implies b\frac{\partial T_1}{\partial y_2} = 0 \implies \frac{\partial T_1}{\partial y_2} = 0$$

So, $\underline{T_1 \text{ is independent of } y_2}$, and $T_1 = T_1(y_1)$ only. As a trial, pick $T_1(y_1) = y_1$. Then

$$\begin{aligned}
x_1 &= y_1 \\
\dot{x}_1 &= \dot{y}_1 = y_2 \qquad \text{(from model)}
\end{aligned}$$

But

$$\dot{x}_1 = x_2 \qquad \text{(linear system)}$$

So

$$\begin{aligned}
x_2 &= \dot{x}_1 = T_2(y_2) = y_2 \\
x_3 &= \dot{x}_2 = T_3(y) = \dot{y}_2 \\
&= -a\sin y_1 - b(y_1 - y_3) \qquad \text{(nonlinear model)} \\
x_4 &= \dot{x}_3 = T_4(y) = -a\dot{y}_1\cos y_1 - b(\dot{y}_1 - \dot{y}_3) \\
&= -ay_2\cos(y_1) - b(y_2 - y_4)
\end{aligned}$$

Finally, as an exercise, check that β and α are well-defined.

Lecture 15

Input-Output Analysis of Nonlinear Systems

15.1. Preliminaries

The understanding of systems from a stimulus-response or input-output or external point of view has a long history, pre-dating the infusion of the state-space or internal descriptions. It is the natural thing to consider in exploring a wide variety of complex systems (from the world of technology to even economics and biology). In some settings, definitions and theorems in the state-space point-of-view lead to corresponding results in the input-output point-of-view. The converse is not case, without additional hypotheses.

To illustrate, consider a linear time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

Assume that

(i) the transition matrix Φ defined such that

$$\begin{aligned} \dot{\Phi}(t, t_0) &= A(t)\Phi(t, t_0) \\ \Phi(t_0, t_0) &= I \end{aligned}$$

satisfies $\|\Phi(t, t_0)\| \leq m \exp(-k(t - t_0))$, $\forall t \geq t_0$, and some $k, m > 0$. (Thus, the system $\dot{x} = A(t)x(t)$ has exponential stability of the zero solution—we call this internal stability.)

(ii) $\|C(t)\| \leq c$ and $\|B(t)\| \leq b$, $\forall t \geq t_0$.

The variation of constants formula tells us that

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma$$

This implies that

$$\|y(t)\| \leq cm\exp(-k(t-t_0))\|x_0\| + \frac{cbm}{k}(1-\exp(-k(t-t_0)))\|u\|$$

where we assume bounded inputs:

$$\|u(t)\| \leq \sup_{t\geq t_0}\|u(t)\| \triangleq \|u\| < \infty$$

(We are careful to distinguish between the norm of vector and the norm of a signal. We have changed from $\|u(t)\|$ which is a norm of the vector u given at time t, and the norm $\|u\|$ which is a norm of the entire signal $u(\cdot)$.)
This implies the final result that

$$\|y\| \leq \beta + \gamma\|u\|$$

where $\beta = cm\|x_0\|$ and $\gamma = \frac{cbm}{k}$. In other words, internal stability and the assumption of bounded inputs implies bounded outputs.

The property of bounded inputs always giving rise to bounded outputs is a type of external stability (as known as Bounded-Input-Bounded-Output (BIBO) stability when the L_∞ norm is used). However, external stability does not imply internal stability, as we can see from the example below.

Example 15.1.

$$\begin{aligned}\dot{x}_1 &= -x_1 + u\\ \dot{x}_2 &= x_1^2 + x_2^2\\ y &= x_1\end{aligned}$$

Here, external stability does NOT imply internal stability of the zero solution. The x_2 dynamics are unstable and are not observed in the output.

We would like to state and prove certain basic notions and theorems of external stability, connect them to interesting physical properties of systems and establish ties to notions of internal stability. The initial steps in this direction include:

(i) proper definitions of function spaces of input and output signals

(ii) concepts of causality, feedback, well-posedness, and passivity

(iii) various stability and finite-gain theorems.

The signals applicable in the present context cannot be of infinite energy over an infinite time interval $[0, \infty)$, (e.g. ramp signals).

Definition 15.1. *The truncation operator $(\cdot)_T$ on functions on $[0, \infty)$ is defined by*

$$x_T(t) = \begin{cases} x(t) & t \leq T \\ 0 & t > T \end{cases}$$

for $T \geq 0$ △

Definition 15.2. *The space L_p is defined by*

$$L_p[0, \infty) = \left\{ x(\cdot) : [0, \infty) \to \mathbb{R} : \int_0^\infty |x(t)|^p dt < \infty \right\}$$

△

Definition 15.3. *The space L_{pe} is defined by*

$$L_{pe} = \{x(\cdot) : [0, \infty) \to \mathbb{R} : x_T \in L_p, \forall T \geq 0\}$$

△

Example 15.2.

$$x(t) = t \qquad t \geq 0$$

$x(\cdot) \notin L_p$ for any $p \in [1, \infty)$. But $x_T(\cdot) \in L_p$, $\forall T \geq 0$.

Lemma 15.1

For each $p \in [1, \infty]$, the set $L_{pe}[0, \infty]$ is a linear space. If $p \in [1, \infty]$ and $f \in L_{pe}[0, \infty)$, then

(i) $\|f_T(\cdot)\|$ is a nondecreasing function of T

(ii) $f \in L_p[0, \infty)$ if and only if there exists a finite constant m such that $\|f_T\| \leq m$, $\forall T > 0$. In that case, $\|f\|_p = \lim_{T \to \infty} \|f_T\|_p$.

□

Exercise 15.1. Prove Lemma 15.1.

Remark 15.1. $L_{pe}[0, \infty)$ itself does not carry a norm. Its norm agrees with the norm on $L_p[0, \infty)$ when restricted to that subspace. △

Let us denote

$$L_p^r = L_p \times L_p \times \ldots \times L_p \qquad (r \text{ times}).$$

That is, each function $f \in L_p^r$ is an $\mathbb{R}^r$ valued function characterized by each component $f_i \in L_p$. We define L_{pe}^r similarly.

Definition 15.4 (Causality). *$F : L_{pe}^m \to L_{pe}^q$ is said to be a causal map/system if*

$$(F(u))_T = (F(u_T))_T \qquad \forall T \geq 0 \quad \textit{and} \quad \forall u \in L_{pe}^m.$$

△

Lemma 15.2

A map/system $F : L_{pe}^m \to L_{pe}^q$ is causal if and only if whenever $u_1, u_2 \in L_{pe}^m$ and $(u_1)_T = (u_2)_T$ for some $T < \infty$, we have $(F(u_1))_T = (F(u_2))_T$. □

Proof of Lemma 15.1

($\Longrightarrow$) Suppose F satisfies the condition in the statement. Let $u \in L_{pe}^m$. Let $T < \infty$, be arbitrary. Then $(u)_T = (u_T)_T$. By hypothesis,

$$(F(u))_T = (F(u_T))_T.$$

Since T is arbitrary, we have established causality.
($\Longleftarrow$) Assume F is causal. Let $u_1, u_2 \in L_{pe}^m$ be such that for some $T > 0$,

$$(u_1)_T = (u_2)_T$$

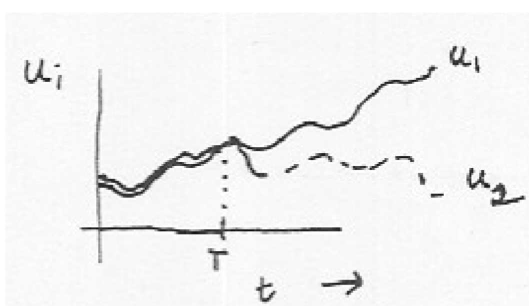


Figure 15.1: Illustration for proof of Lemma.

Now,

$$\begin{aligned}(F(u_1))_T &= (F(u_{1T}))_T \\ &= (F(u_{2T}))_T \\ &= (F(u_2))_T\end{aligned}$$

■

15.2. External Stability

Definition 15.5. *A map/system $F : L_{pe}^m \to L_{pe}^q$ is said to be stable if there exist finite constants $\alpha, \beta > 0$ such that*

$$\|(F(u))_T\| \leq \gamma \|u_T\| + \beta \qquad \forall u \in L_{pe}^m \quad \textit{and} \quad \forall T \geq 0.$$

$\triangle$

Remark 15.2. We refer to the gain as the smallest such γ. We refer to the offset as the smallest such β. $\triangle$

Definition 15.6. *A map/system $F : L_{pe}^m \to L_{pe}^q$ is said to be stable if*

(i) $F(u) \in L_p^q$ whenever $u \in L_p^m$ and in that case,

(ii) there exist constants $\gamma, \beta > 0$ such that

$$\|F(u)\| \leq \gamma \|u\| + \beta \qquad \forall u \in L_p^m$$

$\triangle$

Remark 15.3. The two definitions 15.5 and 15.6 are equivalent, and truncation was not used in definition 15.6. This type of external stability is sometimes referred to as finite-gain L-stability. $\triangle$

15.3. Small Gain Theorem

Theorem 15.1 (*Small Gain Theorem*)

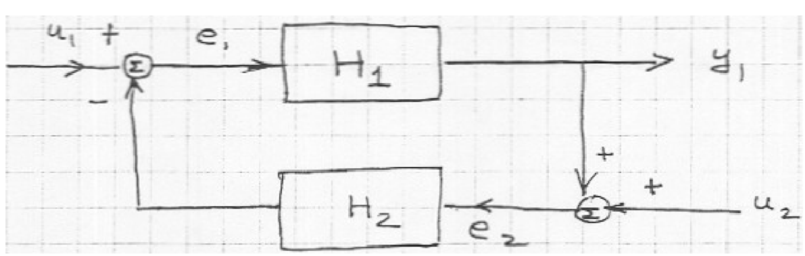


Figure 15.2: Small Gain Theorem illustration.

Assume

(H1) The maps

$$H_1 : L_{pe}^m \to L_{pe}^q$$
$$H_2 : L_{pe}^q \to L_{pe}^m$$

are causal.

(H2) The H_i are stable with gains γ_i and offsets β_i satisfying

$$\|H_i(u)_T\| \leq \gamma_i \|u_T\| + \beta_i \qquad i = 1, 2.$$

(H3) For every pair of inputs $u_1 \in L_{pe}^m$ and $u_2 \in L_{pe}^q$, there exist unique outputs $e_1 \in L_{pe}^m$ and $e_2 \in L_{pe}^q$ (Well-posedness).

If further,

$$\gamma_1 \gamma_2 < 1,$$

then,

(i) $\forall u_1 \in L_{pe}^m$, and $u_2 \in L_{pe}^q$,

$$\|e_{1T}\| \leq \frac{1}{1 - \gamma_1\gamma_2} \left(\|u_{1T}\| + \gamma_2 \|u_{2T}\| + \beta_2 + \gamma_2\beta_1\right)$$
$$\|e_{2T}\| \leq \frac{1}{1 - \gamma_1\gamma_2} \left(\|u_{2T}\| + \gamma_1 \|u_{1T}\| + \beta_1 + \gamma_1\beta_2\right), \qquad \forall T \geq 0, \text{ and}$$

(ii) if $u_1 \in L_p^m$ and $u_2 \in L_p^q$, then $e_1, y_2 \in L_p^m$ and $e_2, y_1 \in L_p^q$, and the norms of e_1 and e_2 are bounded above by the right-hand side in part (i) with nontruncated functions.

□

Proof of Theorem 15.1

Below, we use causality freely to write $\|F(u_T)_T\| = \|F(u)_T\|$ as needed.

By hypothesis (H3), we can solve uniquely for e_{1T} and e_{2T}:

$$e_{1T} = u_{1T} - (H_2(e_{2T}))_T$$
$$e_{2T} = u_{2T} + (H_1(e_{1T}))_T.$$

Then

$$\begin{aligned}
\|e_{1T}\| &\leq \|u_{1T}\| + \|H_2(e_{2T})_T\| \\
&\leq \|u_{1T}\| + \gamma_2 \|e_{2T}\| + \beta_2 \\
&= \|u_{1T}\| + \gamma_2 \|u_{2T} + H_1(e_{1T})_T\| + \beta_2 \\
&\leq \|u_{1T}\| + \gamma_2 \|u_{2T}\| + \gamma_2\gamma_1 \|e_{1T}\| + \gamma_2\beta_1 + \beta_2.
\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$, we can write

$$\|e_{1T}\| \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{1T}\| + \gamma_2\|u_{2T}\| + \beta_2 + \gamma_2\beta_1\right),$$

and similarly for $\|e_{2T}\|$. This completes the proof of part (i).

If $u_1 \in L_p^m$ and $u_2 \in L_p^q$, then,

$$\begin{aligned} \|u_{1T}\| &\leq \|u_1\| \qquad \forall T \geq 0, \quad \text{and} \\ \|u_{2T}\| &\leq \|u_2\| \qquad \forall T \geq 0. \end{aligned}$$

Hence, $\|e_{iT}\|$is bounded uniformly in T. This implies $e_1 \in L_p^m$ and $e_2 \in L_p^q$.

$$\begin{aligned} \|y_{1T}\| &\leq \gamma_1\|e_{1T}\| + \beta_1 \qquad \forall T \geq 0 \\ &\leq \gamma_1\|e_1\| + \beta_1 \qquad \text{uniformly in } T. \end{aligned}$$

This implies $y_1 \in L_p^q$ and by similar arguments, it can be shown that $y_2 \in L_p^m$. ■

Remark 15.4. We interpret the above result in saying that the feedback system is stable if $\gamma_1\gamma_2 < 1$. △

In the Small Gain Theorem, the well-posedness hypothesis (H3) appears to be hard to verify. One would like a sufficient condition that would be strong enough to imply this. The assumption of a stronger hypothesis can ensure that hypothesis (H3) holds in fact.

Definition 15.7. *A map $F : L_{pe}^m \to L_{pe}^q$ is said to be incrementally finite gain stable if*

(i) $F(0) \in L_p^q$ where 0 is the identically zero input.

(ii) For all $T > 0$, $u, v \in L_{pe}^m$, there exists $k > 0$ such that

$$\|F_T(u) - F_T(v)\| \leq k\|u_T - v_T\|$$

(k is independent of T, u, v, etc.)

△

Lemma 15.3

If $F : L_{pe}^m \to L_{pe}^m$ is causal and incrementally finite gain stable with gain $k < 1$, then there is a unique $u^* \in L_{pe}^m$ such that

$$F(u^*) = u^*.$$

□

Proof of Lemma 15.2

By hypothesis,

$$\|F_T(u) - F_T(v)\| \leq k \|u_T - v_T\|,$$

$\forall u, v \in L^m_{pe}$, $\forall T > 0$, *and* $k < 1$.

By causality, $F_T(u) = F_T(u_T)$.

Hence,

$$\|F_T(u_T) - F_T(v_T)\| \leq k \|u_T - v_T\|, \qquad \forall T \geq 0.$$

But,

$$\begin{aligned} \|F(u) - F(v)\| &\leq \sup_{T\geq 0} \|F_T(u) - F_T(v)\| \\ &< k \sup_{T\geq 0} \|u_T - v_T\| \\ &= k \|u - v\| \qquad \forall u, v \in L^m_p \end{aligned}$$

Thus, $F : L^m_p \to L^m_p$ *the restriction to* L^m_p, *is a global contraction. Since* L^m_p *is a Banach space, there is a unique fixed point* $u^* \in L^m_p$ *such that*

$$F(u^*) = u^*.$$

(We can compute u^* *by the successive approximation algorithm initialized in* L^m_p*.)* ■

Exercise 15.2. Can there be a $v^* \in L_{pe}$, but $v^* \notin L_p$ such that $F(v^*) = v^*$ (and $v^* \neq u^*$ necessarily)?

Cleanup the argument in Theorem 4.17 in Sastry.

Example 15.3.

$$\begin{aligned} H : L_{\infty e} &\to L_{\infty e} \\ u &\mapsto u^2 \end{aligned}$$

is causal but unstable.

Example 15.4.

$$H_1(u)(t) = \int_0^t \exp(-a(t-\tau))u(\tau)d\tau$$
$$H_2(u)(t) = ku(t) \qquad a > 0$$

Then,

$$H_1 : L_{\infty e} \to L_{\infty e}$$

with $\gamma_a = \frac{1}{a}$ and $\beta_1 = 0$. Also,

$$H_2 : L_{\infty e} \to L_{\infty e}$$

with $\gamma_2 = |k|$ and $\beta_2 = 0$.
The small gain theorem says that $\frac{1}{a}|k| < 1$ implies stability of the closed loop system. This translates into the requirement that $-a < k < a$. This is conservative in the sense that $-a < k$ is a necessary and sufficient condition for closed loop stability (from the transfer function $g_{CL}(s) = \frac{1}{s+a+k}$).

Lecture 16

Absolute Stability via Lyapunov Theory

This topic originates with the work of Alexander Luré, a Russian mathematician from Leningrad/St. Petersburg. The question can be simply stated as follows.

Given a linear system with a memoryless nonlinear element in the feedback loop, about which we know very little (say that we know it lies in a sector), when can we know that the origin is an asymptotically stable equilibrium for the closed loop system? (See A. I. Luré (1951): Einige nichtlineare problem aus der theorie der automatischen reyeling. Moscow 1951 (R), transl. Berlin (1957).)

Formally, given,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(t, y) \end{aligned} \tag{16.1}$$

(16.2)

where ψ satisfies, for each $t \geq 0$,

$$(\psi(t,y) - K_{\min}y)^T(\psi(t,y) - K_{\max}y) \leq 0 \qquad \text{(sector condition)}, \tag{16.3}$$

for $K_{\min}, K_{\max}$ such that $K = K_{\max} - K_{\min}$ is symmetric and positive definite.

Under what conditions on $G(s) = C(sI - A)^{-1}B$ on $K_{\min}$ and $K_{\max}$, can we conclude that the origin is an asymptotically stable equilibrium for the closed loop system? We will treat this problem in a two step process. First, we restrict to A Hurwitz and $K_{\min} = 0$. Then, we return to the original question. Before we begin, let us introduce a bit of terminology.

The following Lemma can help one to understand the sector condition.

Lemma 16.1

$$\alpha y^2 \leq y\psi(y) \leq \beta y^2 \qquad \alpha \leq \beta$$

is equivalent to

$$(\psi(y) - \alpha y)(\psi(y) - \beta y) \leq 0$$

□

Proof of Lemma 16.1

($\Longrightarrow$) *Suppose* $\alpha y^2 \leq y\psi(y) \leq \beta y^2$.
Then $y(\psi(y) - \beta y) \leq 0$ *and* $y(\psi(y) - \alpha y) \geq 0$.
Multiplying these two inequalities,

$$y^2(\psi(y) - \beta y)(\psi(y) - \alpha y) \leq 0.$$

But $y^2 \geq 0$. *Hence,*

$$(\psi(y) - \beta y)(\psi(y) - \alpha y) \leq 0.$$

($\Longleftarrow$) *Multiply the last inequality by* y^2 *and reverse/retrace the steps.* ■

Remark 16.1. If we consider the scalar case, the scalar conditon

$$(\psi(y) - \alpha y)(\psi(y) - \beta y) \leq 0$$

is shown in the graphical sector condition in Figure 16.1.

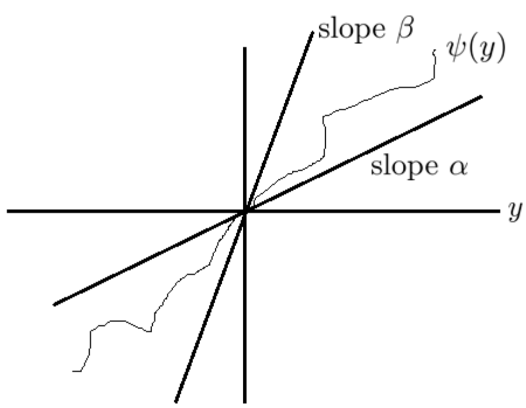


Figure 16.1: Scalar sector condition.

The conditon

$$(\psi(t, y) - K_{\min} y)^T(\psi(t, y) - K_{\max} y) \leq 0$$

is just a multivariate analog of this picture. △

Definition 16.1. *Consider the system*

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\\ u &= -\psi(t, y)\end{aligned}$$

where $t \geq 0$, $\forall y \in \Gamma \subseteq \mathbb{R}^p$, $\mathring{\Gamma}$ *(the interior of* Γ*) connected, includes* 0,

$$(\psi(t, y) - K_{\min} y)^T (\psi(t, y) - K_{\max} y) \leq 0,$$

and $K = K_{\max} - K_{\min} = K^T > 0$. *This system is absolutely stable with a finite domain* Γ *if for the closed-loop system, the origin is uniformly asymptotically stable with any* ψ *satisfying the sector condition.* $\triangle$

If $\Gamma = \mathbb{R}^p$, absolute stability is equivalent to global asymptotic stability.

The main result in the (multivariate) circle criterion. The idea of the proof is to show that under suitable hypotheses, one has a time-independent quadratic Lyapunov function. The key ideas here have to do with the concept of passivity.

Recall that mechanical systems without friction can be cast in the hamiltonian form

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p}\\ \dot{p} &= -\frac{\partial H}{\partial x} + f\end{aligned}$$

Here f is an external (generalized) force corresponding to the generalized coordinate x. Now, we define the rate at which mechanical work is done to the system by the external force f applied to the system as,

$$S = \langle f, \dot{x} \rangle \,. \tag{16.4}$$

We sometimes refer to this rate as the supply rate, S.

Then, if we look at the time rate of change of the hamiltonian, we see

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p}\\ &= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(-\frac{\partial H}{\partial x} + f\right)\\ &= \frac{\partial H}{\partial p} \cdot f\\ &= \dot{x} \cdot f = S\end{aligned}$$

Thus, $\frac{dH}{dt}$ gives the supply rate, also known as the rate of work (= power). The stored energy in the system (i.e. H) increases at a rate equal to the power input.

If there is internal dissipation, then we have

$$\frac{dH}{dt} \leq S, \tag{16.5}$$

the dissipation inequality.

A passive system is one that satisfies the dissipation inequality. Treating forces as inputs (i.e. $u \triangleq f$) and (generalized) velocities as outputs (i.e. $y \triangleq \dot{x}$), we write the dissipation inequality as

$$H(x(t), p(t)) \leq H(x(0), p(0)) + \int_0^t y^T(\sigma)u(\sigma)d\sigma. \tag{16.6}$$

Definition 16.2. *A system is passive if*

$$\int_0^t y(\sigma)^T u(\sigma)d\sigma \geq 0 \qquad \forall t \geq 0. \tag{16.7}$$

$\triangle$

This definition is an abstract one for the general setting of input-output systems. For hamiltonians that are a priori bounded below (say $H(x,p) \geq c$), we see that

$$\begin{aligned} H(x(t), p(t)) - c &\geq 0 \\ H(x(0), p(0)) - c + \int_0^t y^T(\sigma)u(\sigma)d\sigma &\geq \\ \delta + \int_0^t y^T(\sigma)u(\sigma)d\sigma &\geq \end{aligned}$$

which says that

$$\int_0^t y(\sigma)^T u(\sigma)d\sigma$$

is bounded below $\forall t \geq 0$ by a constant ($-\delta$) that depends on the initial conditions.

Definition 16.3. *A $p \times p$ matrix $Z(s)$ of transfer functions is positive real if*

(i) $Z(s)$ *has all matrix elements analytic in* $\{s : \mathrm{Re}(s) \geq 0\}$,

(ii) $Z^*(s) = Z(s^*)$ *for* $\{s : \mathrm{Re}(s) > 0\}$, *and*

(iii) $Z^T(s^*) + Z(s)$ *is positive semidefinite for* $\{s : \mathrm{Re}(s) > 0\}$,

where (*) denotes the complex conjugate and (^{T}) denotes the matrix transpose.

$Z(s)$ is strictly positive real if $Z(s-\epsilon)$ is positive real for some $\epsilon > 0$. △

Remark 16.2. Positive real transfer functions are impedence or admittance matrices made of linear resistors, capacitors, inductors, transformers, and gyrators. △

Lemma 16.2 (*Positive Real Lemma of Kalman-Yacubovitch-Popov*)

Let $Z(s) = C(sI-A)^{-1}B + D$ be a $p \times p$ transfer function of the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned}$$

where A is Hurwitz, $[A, B]$ is controllable, $[A, C]$ is observable. Then, Z is strict positive real if and only if there exists $P = P^T > 0$, matrices W, L, and constant $\epsilon > 0$ such that

$$\begin{aligned} A^T P + PA &= -L^T L - \epsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

□

Proof of Lemma 16.2

(sufficiency)
Suppose there exist P, L, W, ϵ satisfying the above equations. Take $\mu \in (0, \frac{\epsilon}{2})$.

$$(A+\mu I)^T P + P(A+\mu I) = -L^T L - (\epsilon - 2\mu)P,$$

$P > 0$ and $L^T L + (\epsilon - 2\mu)P > 0$. Then by standard matrix Lyapunov theory (Theorem 5.36 of Sastry), the matrix $(A+\mu I)$ is Hurwitz. Hence, $Z(s-\mu)$ is analytic in $\{s : \mathrm{Re}(s) \geq 0\}$.

Let $\Phi = (sI-A)^{-1}$.

$$Z(s-\mu) + Z^T(-s-\mu) = D + D^T + C\Phi(s-\mu)B + B^T\Phi^T(-s-\mu)C^T.$$

Substituting $C = (PB + L^T W)^T$ and $D + D^T = W^T W$, we get,

$$\begin{aligned} &Z(s-\mu) + Z^T(-s-\mu) \\ &= W^T W + (B^T P + W^T L)\Phi(s-\mu)B + B^T\Phi^T(-s-\mu)(PB + L^T W) \\ &= W^T W + W^T L\Phi(s-\mu)B + B^T\Phi^T(-s-\mu)L^T W \\ &\qquad + B^T\Phi^T(-s-\mu)PB + B^T P\Phi(s-\mu)B \\ &\qquad \vdots \end{aligned}$$

■

Exercise 16.1. Complete the remainder of the proof of the Positive Real Lemma.

Theorem 16.1 (*Multivariate Circle Criterion - Hurwitz Case*)

Let $[A, B, C]$ be a controllable and observable triple. Let A be Hurwitz. Suppose ψ satisfies the sector condition,

$$\psi^T(t,y)(\psi(t,y) - Ky) \leq 0$$

$\forall t \geq 0$, $y \in \mathbb{R}^m$, *and* $K = K^T > 0$. *Then the closed-loop system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(t,y), \end{aligned} \tag{16.8}$$

is absolutely stable provided

$$Z(s) = I_m + KG(s)$$

is strict positive real. □

Remark 16.3. If the sector condition is only valid for $\Gamma \subset \mathbb{R}^p$, $0 \in \mathring{\Gamma}$, then the strict positive reality of $Z(s)$ ensures only that the closed-loop system is absolutely stable with finite domain. △

Proof of Theorem 16.1

$Z(s) = I_m + KG(s)$ *is the transfer function of the linear system*

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} \\ \tilde{y} &= KC\tilde{x} + \tilde{u}, \end{aligned}$$

where we have set $D = D^T = I_m$ *in the Positive Real Lemma, and we have further replaced* C *in the lemma by* KC. *One concludes that there exists* $P = P^T > 0$, *matrices* L, W *and constant* $\epsilon > 0$ *such that*

$$\begin{aligned} A^T P + PA &= -L^T L - \epsilon P \\ PB &= (KC)^T - L^T W \\ W^T W &= D + D^T = 2I_m \end{aligned}$$

Take $W = \sqrt{2} I_m \implies PB = C^T K - \sqrt{2} L^T$. *Now consider the function*

$$V(x) = x^T P x.$$

Along trajectories of the closed-loop system,

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax - B\psi(t, Cx))^T P x + x^T P(Ax - B\psi(t, Cx)) \\ &= x^T (A^T P + PA) x - 2x^T PB\psi(t, Cx). \end{aligned}$$

Since $-2\psi^T(t,y)(\psi(t,y) - Ky) \geq 0$ *by the sector condtiion, it follows that*

$$\begin{aligned}
\dot{V} &\leq x^T(A^TP + PA)x - 2x^TPB\psi - 2\psi^T(\psi - KCx) \\
&= x^T(A^TP + PA)x + 2x^T(C^TK - PB)\psi - 2\psi^T\psi \\
&= -\epsilon x^TPx - x^TL^TLx + 2\sqrt{2}x^TL^T\psi - 2\psi^T\psi \qquad (\text{by KYP lemma}) \\
&= -\epsilon x^TPx - (Lx - \sqrt{2}\psi)^T(Lx - \sqrt{2}\psi) \\
&\leq -\epsilon x^TPx.
\end{aligned}$$

The function V *satisfies all the hypotheses of the Time-Varying Lyapunov Theorem with* $\alpha_1(r) = \lambda_{\min}(P)r^2$*,* $\alpha_2 = \lambda_{\max}(P)r^2$*, and* $\alpha_3(r) = \epsilon\lambda_{\min}(P)r^2$*.*

So the closed-loop system has the origin as an uniformly asymptotically stable (in fact exponentially stable) equilibrium point. ■

Remark 16.4. We have not used the easily shown fact that $[A, C]$ observable $\iff$ $[A, KC]$ is observable for any nonsingular matrix K. △

Remark 16.5. Suppose A is not Hurwitz. It is possible that there exists a $K_{\min}$ such that the matrix $[A - BK_{\min}]$ is Hurwitz (under the assumption that $[A, B]$ is controllable and $[A, C]$ is observable). (In fact, conditions for this are difficult to determine and there is a deep problem hidden here— see the work of Byrnes, Brockett, Rosenthal, and others. This work involves the methods of algebraic geometry, including the Schubert calculus. We will sweep these difficulties under the rug!) △

Now the closed loop system of

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx \\
u &= -\psi(t, y)
\end{aligned}$$

is given by

$$\begin{aligned}
\dot{x} &= Ax - B\psi(t, Cx) \\
&= (A - BK_{\min}C)x - B(\psi(t, Cx) - K_{\min}Cx)
\end{aligned}$$

which is the closed loop system of

$$\begin{aligned}
\dot{x} &= (A - BK_{\min}C)x + Bu \\
y &= Cx \\
u &= -\tilde{\psi}(t, y),
\end{aligned}$$

where

$$\tilde{\psi} \triangleq \psi(t, y) - K_{\min}y.$$

Note that the triple $[A, B, C]$ is controllable and observable if and only if the triple $[A - BK_{\min}C, B, C]$ is controllable and observable. (This is only an exercise in linear algebra).

The transfer function $\tilde{G}(s)$ of the system

$$\begin{aligned} \dot{z} &= (A - BK_{\min}C)z + Bu \\ y &= Cz \end{aligned}$$

is the same as the transfer function of the closed-loop system in Figure 16.2, where $G(s) = C(sI - A)^{-1}B$ as before.

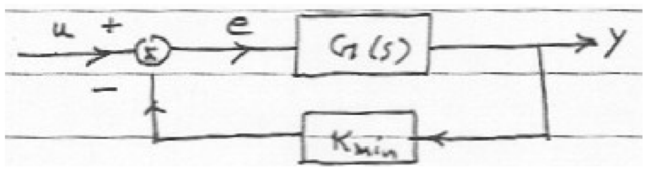


Figure 16.2: Closed-loop system.

Observe that in terms of Laplace transforms of inputs and outputs,

$$\begin{aligned} Y(s) &= G(s)E(s) \\ E(s) &= U(s) - K_{\min}Y(s) \end{aligned}$$

This implies

$$\begin{aligned} Y(s) &= (I + G(s)K_{\min})^{-1}G(s)U(s), \\ \tilde{G}(s) &= (I + G(s)K_{\min})^{-1}G(s). \end{aligned}$$

On the other hand,

$$\begin{aligned} E(s) &= U(s) - K_{\min}Y(s) \\ &= (I + K_{\min}G(s))^{-1}U(s). \end{aligned}$$

This implies,

$$\begin{aligned} Y(s) &= G(s)E(s) \\ &= G(s)(I + K_{\min}G(s))^{-1}U(S). \end{aligned}$$

And so we have the equivalent formulas (can be shown with a little algebra):

$$\begin{aligned} \tilde{G}(s) &= (I + G(s)K_{\min})^{-1}G(s) && (16.9) \\ &= G(s)(I + K_{\min}G(s))^{-1} && (16.10) \end{aligned}$$

Now $A - BK_{\min}C$ is Hurwitz if and only if all the poles of $\tilde{G}(s)$ are in $\mathbb{C}^{-}$. Applying the sector condtiion to $\tilde{\psi}$,

$$\tilde{\psi}(t,y)^T(\tilde{\psi}(t,y) - Ky) \leq 0$$

$$\iff \quad (\psi(t,y) - K_{\min}y)^T(\psi(t,y) - (K_{\min} + K)y) \leq 0$$

$$\iff \quad (\psi(t,y) - K_{\min}y)^T(\psi(t,y) - K_{\max}y) \leq 0$$

for $K_{\max} = K_{\min} + K$. The relevant positive real transfer function is

$$\begin{aligned}
\tilde{Z}(s) &= I + K\tilde{G}(s) \\
&= I + KG(s)(I + K_{\min}G)^{-1} \\
&= (I + K_{\min}G)(I + K_{\min}G)^{-1} + KG(s)(I + K_{\min}G)^{-1} \\
&= (I + (K_{\min} + K)G)(I + K_{\min}G)^{-1} \\
&= (I + K_{\max}G)(I + K_{\min}G)^{-1}.
\end{aligned}$$

Now we are ready to state the multivariate circle criterion without the Hurwitz assumption.

Theorem 16.2 (*Multivariate Circle Criterion*)

Let $[A, B, C]$ be a controllable and observable triple. Suppose ψ satisfies the sector condition,

$$(\psi^T(t,y) - K_{\min}y)^T(\psi(t,y) - K_{\max}y) \leq 0$$

$\forall t \geq 0$, $y \in \mathbb{R}^m$, and $K = K_{\max} - K_{\min} = K^T > 0$ given.

Then the closed-loop system

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx \\
u &= -\psi(t,y),
\end{aligned} \tag{16.11}$$

is absolutely stable provided

(i) $\tilde{G}(s) = G(s)(I + K_{\min}G(s))^{-1}$ is "Hurwitz" (analytic in $\{s : \mathrm{Re}s \geq 0\}$), and

(ii) $\tilde{Z}(s) = (I + K_{\max}G)(I + K_{\min}G)^{-1}$ is strict positive real.

□

Proof of Theorem 16.2

From the remarks preceding the above statement, it is clear that all one has to do it to appeal to the equivalences of closed loop systems with and without loop transformation arising from the feedback $A \mapsto A - BK_{\min}C$ and appeal to the

Hurwitz case already proved. ■

Where does the name "Circle Criterion" come from? This is an interesting story going back to the work of Harry Nyquist[1], the AT&T Mathematician who investigated graphical methods for feedback amplifier stability in long-distance (transatlantic) telephony. This is a direct application of the principle of the argument in complex variable theory.

First we specialize to the single input, single output case. Let

$$\begin{aligned}\Gamma_G &\triangleq \{u + jv = G(j\omega) : \omega \in \mathbb{R}\} \\ &= \text{ image of the imaginary axis under } G(\cdot),\end{aligned}$$

be the Nyquist locus of G.

Theorem 16.3 (*Nyquist*)

Let Γ_G be bounded (i.e. G is proper and has no poles on the $j\omega$ axis), We will say that Γ_G encircles a point u_0+jv_0, ρ times, if u_0+jv_0 is not on Γ_G and $2\pi\rho =$ the net increase in the argument of $G(j\omega) - (u_0 + jv_0)$ as ω increases from $-\infty$ to $+\infty$. Clockwise encirclement corresponds with the direction of increasing argument, while counterclockwise encirclement corresponds with the direction of decreasing argument.

Suppose Γ_G is bounded. If G has ν poles in the right half plane $\mathbb{C}^+$, then $\frac{G}{1+kG}$ has $\rho + \nu$ poles in $\mathbb{C}^+$ if the point $-\frac{1}{k} + j0$ is not on Γ_G and Γ_G encircles it ρ times in the clockwise sense. □

Proof of Theorem 16.3

See Franklin et al. listed in the reference list. ■

Corollary 16.1

If Γ_G is bounded and $-\frac{1}{k} + j0$ is not on Γ_G and G has ν poles in $\mathbb{C}^+$, then the feedback $u = -ky$ stabilizes the closed loop system if Γ_G encircles $(-\frac{1}{k} + j0)$, ν times in the counterclockwise direction. □

Lemma 16.3

Let $g(s)$ be a scalar transfer function. Let $g(s)$ be proper (i.e. $g(s) = \frac{q(s)}{p(s)} + d$ where $\deg(q) < \deg(p)$, p monic, and d a constant). Suppose poles of $g(s)$ all lie in $\mathbb{C}^-$, then $g(s)$ is strict positive real if and only if

$$\mathrm{Re}(g(j\omega)) > 0 \qquad \forall\omega \in \mathbb{R}.$$

□

[1] Nyquist is the same man who developed the theoretical understanding of thermal noise (Nyquist-Johnson noise) in electrical devices, in parallel with the work of experimentalist Johnson (in 1928). This understanding has become extremely important today, in the context of MEMS devices.

Proof of Lemma 16.3

See H. Khalil page 404. ■

Theorem 16.4

Let $g(s)$ be a scalar transfer function $c(sI - A)^{-1}b$, with the triple [A,b,c] controllable and observable. Let $\psi(t, y)$ satisfy the sector condition:

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

Then absolute stability of the closed loop system

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \\ u &= -\psi(t, y) \end{aligned}$$

holds provided one of the following conditions apply:

(i) If $0 < \alpha < \beta$, the Nyquist locus does not enter the disk $D(\alpha, \beta)$ and encircles it ν times in the counterclockwise direction, where ν = # poles of $g(s)$ in $\mathbb{C}^+$.

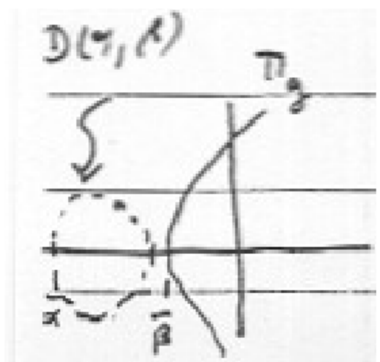


Figure 16.3: Example of Nyquist locus.

(ii) If $0 = \alpha < \beta$, $g(s)$ is "Hurwitz", and the Nyquist plot Γ_g lies to the right of the line $\mathrm{Re}(s) = -\frac{1}{\beta}$

(iii) If $\alpha < 0 < \beta$, $g(s)$ is "Hurwitz", and the Nyquist plot of Γ_g lies in the interior of the disk $D(\alpha, \beta)$.

□

Proof of Theorem 16.4

Specializing the multivariable circle criterion to this case, we seek conditions to ensure that

(a) $\frac{g(s)}{1+\alpha g(s)}$ *is Hurwitz and*

(b) $\frac{1+\beta g(s)}{1+\alpha g(s)}$ *is strict positive real.*

For (b) it is equivalent to check

$$\mathrm{Re}\left(\frac{1+\beta g(j\omega)}{1+\alpha g(j\omega)}\right) > 0 \qquad \forall \omega \in \mathbb{R}$$

In case (i), for $0 < \alpha < \beta$*, this is equivalent to checking*

$$\mathrm{Re}\left(\frac{\frac{1}{\beta}+g(j\omega)}{\frac{1}{\alpha}+g(j\omega)}\right) > 0 \qquad \forall \omega \in \mathbb{R}.$$

Consider Figure 16.4.

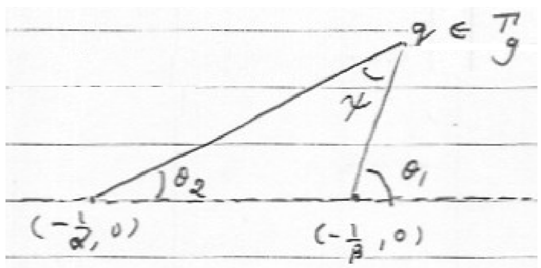


Figure 16.4: Illustration for proof.

$$\begin{aligned}
\psi &= \theta_1 - \theta_2 \\
\theta_1 &= \arg\left(\frac{1}{\beta} + g(j\omega)\right) \\
\theta_2 &= \arg\left(\frac{1}{\alpha} + g(j\omega)\right) \\
\mathrm{Re}\left(\frac{\frac{1}{\beta}+g(j\omega)}{\frac{1}{\alpha}+g(j\omega)}\right) &= r\cos\psi
\end{aligned}$$

where $r > 0$, $\cos\psi > 0$ *if and only if* $\psi = \theta_1 - \theta_2 < \frac{\pi}{2}$.

By elementary geometry, q *has to lie outside* $D(\alpha, \beta)$ *the disk with diameter joining* $\left(-\frac{1}{\alpha}, 0\right)$ *and* $\left(-\frac{1}{\beta}, 0\right)$*. For the encirclement condition, we use the corollary to Nyquist.*

In case (ii), the condition for strict positive reality is

$$\begin{aligned}
&\mathrm{Re}\left(\frac{1}{\beta} + g(j\omega)\right) > 0 \\
\iff \quad &\cos\theta_1 > 0 \\
\iff \quad &\theta_1 < \frac{\pi}{2} \\
\iff \quad &\Gamma_g \text{ lies to the right of the vertical line } \left\{s : \mathrm{Re}(s) = -\frac{1}{\beta}\right\}.
\end{aligned}$$

In case (iii), the same arguments as in (i) apply, but since α and β have opposite sign, the strict positive reality condition is

$$\mathrm{Re}\left(\frac{\frac{1}{\beta}+g(j\omega)}{\frac{1}{\alpha}+g(j\omega)}\right)<0,$$

and we seek $\psi > \frac{\pi}{2}$. This occurs if and only if Γ_g lies in the interior of the disk $D(\alpha, \beta)$. ■

The Popov criterion for absolute stability is based on

(i) restrictions on ψ,
$\psi = \psi(y) = (\psi_1(y_1), \psi_2(y_2), \dots, \psi_m(y_m))^T$
$\psi^T(y)(\psi(y) - Ky) \leq 0$
$K = \mathrm{diag}(\beta_1, \dots, \beta_m) \qquad \beta_i \geq 0, \forall i,$
and

(ii) use of nonquadratic Lyapunov functions,

$$\begin{aligned} V(x) &= x^T P x + 2\eta \int_0^y \sum_{i=1}^m \psi_i(\sigma_i)\beta_i d\sigma_i \\ &\triangleq x^T P x + 2\eta \int_0^y \psi^T(\sigma) K d\sigma \qquad \text{with } y = Cx. \end{aligned}$$

By the sector condition, $\psi(\sigma) \geq 0$ for $\sigma \geq 0$ (componentwise) implies

$$\int_0^y \sum_{i=1}^m \psi_i(\sigma_i)\beta_i d\sigma_i \geq 0$$

This further implies

$$V(x) > 0 \qquad \text{(as long as } P = P^T > 0).$$

Along trajectories of the (usual) closed loop system

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} + 2\eta \psi^T K \dot{y} \\ &= (AX - B\psi)^T P x + x^T P (A - B\psi) + 2\eta \psi^T K C (Ax - B\psi) \\ &= x^T (A^T P + PA) x - 2x^T P B \psi + 2\eta \psi^T K C (Ax - B\psi) \end{aligned}$$

Since $-2\psi^T(\psi - Ky) \geq 0$, we get,

$$\begin{aligned} \dot{V} &\leq x^T (A^T P + PA) x - 2x^T P B \psi + 2\eta \psi^T K C (Ax - B\psi) - 2\psi^T(\psi - Ky) \\ &= x^T (A^T P + PA) x - 2x^T (PB - \eta^T A^T C^T K - C^T K)\psi \\ &\quad - \psi^T (2I + \eta K C B + \eta B^T C^T K)\psi \end{aligned}$$

Choose η small enough such that

$$2I + \eta KCB + \eta B^T C^T K \geq 0$$

if and only if we can find W such that

$$\begin{aligned} W^T W &= wI + \eta KCB + \eta B^T C^T K \\ &= (I + \eta KCB) + (I + \eta KCB)^T \end{aligned}$$

Suppose $P = P^T > 0$ and there exists L and $\epsilon > 0$ such that

$$\begin{aligned} A^T P + PA &= -L^T L - \epsilon P \\ PB &= C^T K + \eta A^T C^T K - L^T W \\ &= (C + \eta CA)^T K - L^T W \end{aligned}$$

Then,

$$\begin{aligned} \dot{V}(x) &\leq -\epsilon x^T P x - x^T L^T L x + 2x^T L^T W \psi - \psi^T W^T W \psi \\ &= -\epsilon x^T P x - (Lx - W\psi)^T (Lx - W\psi) \\ &\leq \epsilon x^T P x \\ &< 0 \qquad x \neq 0. \end{aligned}$$

Thus, we get absolute stability. The question of P, L, ϵ, W is settled by the Positive Real Lemma.

$$\begin{aligned} Z(s) &= (I + \eta KCB) + (KC + \eta KCA)(sI - A)^{-1} B \\ &= I + \eta KC(sI - A + A)(sI - A)^{-1} B + KC(sI - A)^{-1} B \\ &= I + \eta s KC(sI - A)^{-1} B + KC(sI - A)^{-1} B \\ &= I + (\eta s + 1) KG(s) \end{aligned}$$

Suppose η is chosen that $-\frac{1}{\eta}$ is not an eignevalue of A. Then $[A, K(C + \eta CA)]$ is observable if and only if $[A, C]$ is observable.

Then by the Positive Real Lemma, P, L, ϵ, W exist if $Z(s) = I + (\eta s + 1) KG(s)$ is strict positive real, which we have proved.

Theorem 16.5 (*Multivariate Popov Criterion*)

For the closed-loop system,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y), \end{aligned} \tag{16.12}$$

let $[A, B, C]$ be a controllable and observable triple. Suppose A is Hurwitz, $\psi = \psi(y) = (\psi_1(y_1), \psi_2(y_2), \ldots, \psi_m(y_m))^T$, and $K = \text{diag}(\beta_1, \ldots, \beta_m)$, $\beta_i \geq 0, \forall i$. Suppose further that ψ satisfies the sector condition,

$$0 \leq y_i \psi_i(y_i) \leq \beta_i y_i^2.$$

Then, the closed-loop system is absolutely stable if there exists $\eta \geq 0$ such that

(i) $-\frac{1}{\eta} \in$ *spec*(A), *and*

(ii) $Z(s) = I_m + (1 + \eta s) K G(s)$ *is strict positive real.*

□

Remark 16.6. (a) With $\eta = 0$, this reduces to a special case of the circle criterion.

(b) With $\eta > 0$, we get absolute stability under weaker conditions (but for a restricted class of nonlinear maps ψ).

(c) For $m = 1$ (SISO case), we have a graphical test:

Choose η such that $Z(\infty) = \lim_{s \to \infty} Z(s) = W^2 > 0$.
Then $Z(s)$ is strict positive real if and only if

$$\text{Re}[1 + (1 + \eta j\omega)] > 0 \qquad \forall \omega \in \mathbb{R} \quad (k > 0).$$

This holds if and only if

$$\frac{1}{k} + \text{Re}(g(j\omega)) - \eta\omega \text{Im}(g(j\omega)) > 0 \qquad \forall \omega \in \mathbb{R}.$$

This hold if and only if the Popov locus/plot lies to the right of the line that intercepts the point $-\frac{1}{k} + j0$ with slope η.

Here the Popov locus is given by

$$\mathcal{P}_g = \{u + jv : \quad u = \text{Re}(g(j\omega)), \quad v = \omega \text{Im}(g(j\omega)) \qquad \forall \omega \in \mathbb{R}\}. \quad (16.13)$$

See Figure 16.5 for an illustration.

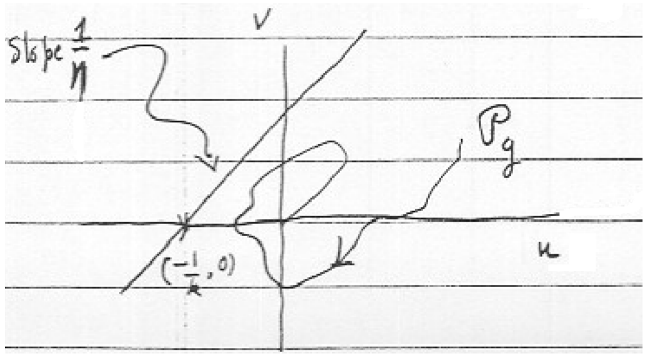


Figure 16.5: Popov locus illustration.

△

Supplements

Supplement A

An Alternate Way to Frame a Curve

A.1. The Natural Frenet Frame

A defect of the Frenet-Serret frame is the need for nondegeneracy ($\kappa > 0$). There is an alternative way to frame a curve without requiring nondegeneracy. This frame is known as the Natural Frenet frame or the Relatively Parallel Adapted Frame (RPAF) and was popularized by R. L. Bishop (ref: R. L. Bishop, American Math. Monthly, 82 (3): 246-251, 1975).

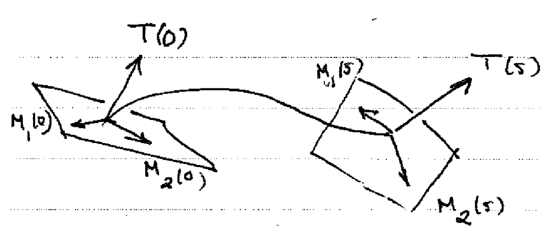


Figure A.1: Propagation of a Natural Frenet frame.

Let s denote an arc-length parameter. At $s = 0$, let $T^{\perp}(0)$ denote the plane normal to the unit tangent vector $T(0)$; similarly, $T^{\perp}(s)$ is the normal plane at s. Pick a basis $\{M_1(0), M_2(0)\}$ for $T^{\perp}(0)$ such that $\{T(0), M_1(0), M_2(0)\}$ constitutes a right handed orthonormal triad. Our goal is to propagate this triad to $\{T(s), M_1(s), M_2(s)\}$ in such a way that certain natural conditions are satisfied:

1. *Right handedness*: $\longleftrightarrow$ $M_2(0) = T(0) \times M_1(0)$, and $M_2(s) = T(s) \times M_1(s)$

2. *Existence of natural curvatures*: $T(s) \cdot T(s) = 1 \implies T'(s) \in T^{\perp}(s)$. Hence, there must exist $k_1(s)$ and $k_2(s)$ such that $T'(s) = k_1(s)M_1(s) + k_2(s)M_2(s)$. We call $k_1(s)$ and $k_2(s)$ (natural) curvatures.

3. $M_1(s)$ and $M_2(s)$ must be relatively parallel fields (see the following definition).

Definition A.1. *Let $M(s)$ be any unit normal field along γ. We say $M(s) \in T^{\perp}(s)$ is a relatively parallel field along γ provided*
*$M'(s) = f(s)T(s)$, i.e. the vector **M(s)** turns as little as possible.* △

We propagate $M_1(0), M_2(0)$ along γ such that they remain relatively parallel to each s. Thus,

$$M_1'(s) = f_1(s)T(s)$$
$$M_2'(s) = f_2(s)T(s)$$

for some as yet undetermined f_1, f_2.

However, by normality, $M_1(s) \cdot T(s) \equiv 0$. This implies

$$\begin{aligned} M_1'(s) \cdot T(s) &= -M_1(S) \cdot T'(s) \qquad \text{(differentiation by } s\text{)} \\ &= -M_1(s) \cdot (k_1(s)M_1(s) + k_2(s)M_2(s)) \\ &= -k_1(s) \end{aligned}$$

Therefore, we see that $f_1(s)$ and $f_2(s)$ are given by $-k_1(s)$ and $-k_2(s)$, respectively.

Definition A.2. *An orthonormal triad $\{T(s), M_1(s), M_2(s)\}$ is a relatively parallel adapted frame (RPAF) along γ provided there exist curvature functions $k_1(\cdot)$ and $k_2(\cdot)$ such that*

$$\begin{aligned} T' &= k_1M_1 + k_2M_2 \\ M_1' &= -k_1T \\ M_2' &= -k_2T \end{aligned}$$

△

We also refer to RPAF's as natural Frenet frames.

Theorem A.1 (*Unique RPAF Along a Curve*)

Given a C^2 curve γ ands a choice $M_1(0), M_2(0)$ in $T^{\perp}(0)$ such that $\{T(0), M_1(0), M_2(0)\}$

is a right-handed orthonormal triad, there is a unique RPAF along γ that agrees with the initial choice. □

Proof of Theorem A.1

Integrating $M_1'(s) = -k_1(s)T(s)$ on both sides,

$$M_1(s) = M_1(0) - \int_0^s k_1(\sigma)T(\sigma)d\sigma$$

Taking

$$T'(s) = k_1(s)M_1(s) + k_2(s)M_2(s),$$

and dotting both sides with $M_1(s)$, we obtain

$$\begin{aligned} k_1(s) &= T'(s) \cdot M_1(s) \\ &= T'(s) \cdot M_1(0) - \int_0^s k_1(\sigma)T'(s) \cdot T(\sigma)d\sigma \\ &= \gamma''(s) \cdot M_1(0) - \int_0^s k_1(\sigma)\gamma''(s) \cdot \gamma'(\sigma)d\sigma \end{aligned}$$

Similarly,

$$k_2(s) = \gamma''(s) \cdot M_2(0) - \int_0^s k_2(\sigma)\gamma''(s) \cdot \gamma'(\sigma)d\sigma$$

Given the curve γ, we have two integral equations for k_1, k_2. By the standard theory of Volterra integral equations, there exist unique k_1, k_2.

Now we may integrate,

$$\frac{d}{ds}\left[\begin{array}{ccc} T(s) & M_1(s) & M_2(s) \end{array}\right] = \left[\begin{array}{ccc} T(s) & M_1(s) & M_2(s) \end{array}\right] \begin{bmatrix} 0 & -k_1(s) & -k_2(s) \\ k_1(s) & 0 & 0 \\ k_2(s) & 0 & 0 \end{bmatrix}$$

Starting from $\left[\begin{array}{ccc} T(0) & M_1(0) & M_2(0) \end{array}\right] \in \mathbb{SO}(3)$, to obtain a unique RPAF. ■

A.2. Relation to the Frenet-Serret Frame

The normal $N(s)$ and binormal $B(s)$, when defined, exist in the plane $T^\perp(s)$, spanned by $M_1(s)$ and $M_2(s)$.

$$\begin{aligned} N(s) &= \frac{1}{\kappa(s)}T'(s) \\ &= \frac{1}{\kappa(s)}\left[k_1(s)M_1(s) + k_2(s)M_2(s)\right] \end{aligned}$$

Since $N(s) \cdot N(s) \equiv 1$,

$$\begin{aligned} N(s) \cdot N(s) &= \frac{k_1^2(s) + k_2^2(s)}{\kappa^2(s)} \\ \implies \kappa(s) &= \sqrt{k_1^2(s) + k_2^2(s)} \end{aligned}$$

We can express the binormal as

$$\begin{aligned} B(s) &= T(s) \times N(s) \\ &= T(s) \times \left(\frac{k_1}{\kappa(s)} M_1(s) + \frac{k_2(s)}{\kappa(s)} M_2(s) \right) \\ &= \frac{1}{\kappa(s)} \left[-k_2(s) M_1(s) + k_1(s) M_2(s) \right], \end{aligned}$$

and use it to find the torsion (suppressing the s dependence),

$$\begin{aligned} \tau(s) &= -B' \cdot N \\ &= -\left(-\frac{k_2}{\kappa} M_1 + \frac{k_1}{\kappa} M_2 \right)' \cdot \left(\frac{k_1}{\kappa} M_1 + \frac{k_2}{\kappa} M_2 \right) \\ &= \left(\frac{k_2'}{\kappa} M_1 - \frac{k_1'}{\kappa} M_2 + \frac{k_2}{\kappa} M_1' - \frac{k_1}{\kappa} M_2' + k_2 M_1 \left(\frac{1}{\kappa} \right)' - k_1 M_2 \left(\frac{1}{\kappa} \right)' \right) \cdot \left(\frac{k_1}{\kappa} M_1 + \frac{k_2}{\kappa} M_2 \right) \\ &= \frac{k_2' k_1 - k_1' k_2}{\kappa^2} \\ &= \frac{k_2' k_1 - k_1' k_2}{k_1^2 + k_2^2} \\ &= \left(\tan^{-1} \left(\frac{k_2}{k_1} \right) \right)' \\ &= \theta' \end{aligned}$$

where θ is a polar angle in the (k_1, k_2) plane (well-defined for $\kappa > 0$). The (k_1, k_2) plane is referred to as the plane of normal development. Note that (κ, θ) represents polar coordinates in the plane of normal development. Further, integrating,

$$\theta(s) = \theta(0) + \int_0^s \tau(\sigma) d\sigma$$

Since

$$\begin{aligned} N(s) &= \cos\theta(s) M_1(s) + \sin\theta(s) M_2(s) \\ B(s) &= -\sin\theta(s) M_1(s) + \cos\theta(s) M_2(s), \end{aligned}$$

it is clear that $\theta(s)$ is the accumulated rotation (phase-shift) of $\{N(s), B(s)\}$ relative to $\{M_1(s), M_2(s)\}$. □

Definition A.3. *One gets a picture of a curve γ in $\mathbb{R}^3$ by looking at its* <u>*normal development*</u> $s \mapsto (k_1(s), k_2(s))$*, the curve traced out in the plane of normal development.* $\triangle$

Example A.1. Show that the normal development of a curve γ constrained to lie in plane $\mu^\perp$ that is perpendicular to vector μ is a line passing through the origin. (We not assume that $\mu^\perp$ passes through the origin.)

$$\begin{aligned} \gamma(s)\cdot\mu &\equiv c \quad \text{a constant} \\ \implies T(s)\cdot\mu &\equiv 0 \\ \implies T'(s)\cdot\mu &\equiv 0 \end{aligned}$$

From the last equation,

$$k_1(s)(M_1(s)\cdot\mu) + k_2(s)(M_2(s)\cdot\mu) \equiv 0$$

On the other hand,

$$\begin{aligned} M_1'(s)\cdot\mu &= -k_1(s)T(s)\cdot\mu \\ &\equiv 0 \\ \implies M_1(s)\cdot\mu &\equiv \text{constant} \triangleq a_1 \end{aligned}$$

Similarly,

$$\implies M_2(s)\cdot\mu \equiv \text{constant} \triangleq a_2$$

Thus the normal development satisfies the constraint of a line passing through the origin,

$$a_1k_1 + a_2k_2 = 0.$$

Note that the normal development contains no information regarding c.

Example A.2. Show that the normal development of a curve $s \mapsto \gamma(s)$, constrained to lie on the surface of a sphere of radius $R > 0$ and center located at p, is a line not passing through the origin.
To lie of the surface of the sphere,

$$(\gamma(s)-p)\cdot(\gamma(s)-p) \equiv R^2.$$

Differentiating,

$$\begin{aligned}(\gamma(s)-p)\cdot\gamma'(s)&\equiv 0\\(\gamma(s)-p)\cdot T(s)&\equiv 0\end{aligned}$$

Differentiating again,

$$\begin{aligned}\gamma'(s)\cdot T(s)+(\gamma(s)-p)\cdot T(s)&\equiv 0\\(\gamma(s)-p)\cdot T(s)&\equiv -1\end{aligned}$$

Dotting both sides of the $T'(s)$ equation with $(\gamma(s)-p)$,

$$k_1(\gamma(s)-p)\cdot M_1+k_2(\gamma(s)-p)\cdot M_2=-1$$

We can also dot both sides of the $M_i'(s)$ equations with $(\gamma(s)-p)$ for $i=1,2$,

$$\begin{aligned}M_i'\cdot(\gamma(s)-p)&=-k_iT\cdot(\gamma(s)-p)\\&\equiv 0\end{aligned}$$

Now,

$$\begin{aligned}(M_i\cdot(\gamma(s)-p))'&=M_i'\cdot(\gamma(s)-p)+M_i\cdot\gamma'(s)\\&\equiv 0\end{aligned}$$

Thus,

$$M_i\cdot(\gamma(s)-p)\equiv\text{constant}\triangleq a_i\qquad i=1,2.$$

Thus, from the expression for $T'(s)$,

$$a_1k_1+a_2k_2\equiv -1.$$

So, we have shown the normal development lies on a line not passing through the origin. To find the distance of the line from the origin, expand $(\gamma(s)-p)$,

$$\begin{aligned}(\gamma(s)-p)&=((\gamma(s)-p)\cdot M_1)M_1+((\gamma(s)-p)\cdot M_2)M_2+((\gamma(s)-p)\cdot T)T\\&=a_1M_1+a_2M_2\end{aligned}$$

Recalling the constraint,

$$\begin{aligned}R^2&=(\gamma(s)-p)\cdot(\gamma(s)-p)\\&=a_1^2+a_2^2.\end{aligned}$$

Therefore, the line lies at a distance of $1/R$ from the origin.
The normal development, being fully Euclidean invariant, contains no information about the center p of the sphere.

Also, note as $R \longrightarrow \infty$, the sphere $\longrightarrow$ a plane, and the above line $\longrightarrow$ a line passing through the origin.

Remark A.1. Recall that the RPAF is determined up to a choice of initial orthonormal basis $\{M_1(0), M_2(0)\}$. A change of basis is simply a rotation of $\{M_1(0), M_2(0)\}$ through an angle ϕ. How does this affect the curvatures k_1 and k_2?
Let

$$\begin{aligned} \tilde{M}_1(0) &= \cos\phi M_1 - \sin\phi M_2 \\ \tilde{M}_2(0) &= \sin\phi M_1 + \cos\phi M_2 \end{aligned}$$

Then it can be shown by substitution into the Volterra equations for the curvatures that

$$\begin{aligned} \tilde{k}_1(s) &= \cos\phi k_1(s) - \sin\phi k_2(s) \\ \tilde{k}_2(s) &= \sin\phi k_1(s) + \cos\phi k_2(s) \end{aligned}$$

This corresponds to a rotation by ϕ in the plane of normal development. A curve γ determines the normal development up to such a rotation. △

Example A.3. Let $s \mapsto \gamma(s)$ be a curve confined to a sphere of radius R centered at $p \in \mathbb{R}^3$. For $s = 0$, pick $M_1(0) = \frac{\gamma(0)-p}{R}$. Find the normal development equation.

Clearly, $M_1(0) \in T^{\perp}(0)$ by hypothesis. We also recall that $(\gamma(s) - p) \cdot (\gamma(s) - p) = R^2$.
We set $M_2(0) = T(0) \times M_1(0)$ to make up the initial orthonormal frame

$$\{M_1(0), M_2(0), T(0)\}.$$

Now, from the previous example for a sphere,

$$M_i \cdot (\gamma(s) - p) \equiv \text{constant} \triangleq a_i \qquad i = 1, 2.$$

So,

$$\begin{aligned}
a_1 &= (\gamma(s) - p) \cdot M_1(s) \\
&= (\gamma(0) - p) \cdot M_1(0) \\
&= (\gamma(0) - p) \cdot \frac{(\gamma(0) - p)}{R} \\
&= R \\
a_2 &= (\gamma(s) - p) \cdot M_2(s) \\
&= (\gamma(0) - p) \cdot M_2(0) \\
&= (\gamma(0) - p) \cdot \left(T(0) \times \frac{(\gamma(0) - p)}{R} \right) \\
&= 0
\end{aligned}$$

The normal development equation becomes,

$$\begin{aligned}
a_1 k_1(s) + a_2 k_2(s) &= -1 \\
R k_1(s) + 0 &= -1 \\
\implies k_1(s) &= -\frac{1}{R}
\end{aligned}$$

This is a vertical line in the plane of normal development.

Thus, the evolution equation of an RPAF for a curve confined to a sphere of radius R (centered at p) can always be taken to be of the form:

$$\frac{d}{ds} \left[\begin{array}{ccc} T(s) & M_1(s) & M_2(s) \end{array} \right] = \left[\begin{array}{ccc} T(s) & M_1(s) & M_2(s) \end{array} \right] \left[\begin{array}{ccc} 0 & 1/R & -k_2(s) \\ -1/R & 0 & 0 \\ k_2(s) & 0 & 0 \end{array} \right]$$

with

$$M_1(0) = \frac{\gamma(0) - p}{R} \qquad M_2(0) = T(0) \times M_1(0).$$

Since

$$\begin{aligned}
(\gamma(s) - p) \cdot M_1(s) &\equiv R, \\
\frac{\gamma(s) - p}{R} \cdot M_1(s) &\equiv 1.
\end{aligned}$$

Let

$$M_1(s) = \frac{\gamma(s) - p}{R} + \delta(s).$$

Then

$$\begin{aligned}\frac{\gamma(s)-p}{R}\cdot\frac{\gamma(s)-p}{R}+\frac{\gamma(s)-p}{R}\cdot\delta(s)&\equiv 1\\ 1+\frac{\gamma(s)-p}{R}\cdot\delta(s)&\equiv 1\\ \frac{\gamma(s)-p}{R}\cdot\delta(s)&\equiv 0\end{aligned}$$

Then

$$\begin{aligned}M_1(s)\cdot M_1(s)&=\left(\frac{\gamma(s)-p}{R}+\delta(s)\right)\cdot\left(\frac{\gamma(s)-p}{R}+\delta(s)\right)\\&=1+\delta(s)\cdot\delta(s).\end{aligned}$$

But $M_1(s)$ is a unit vector for each s. Hence $\delta(s)\cdot\delta(s)\equiv 0$ implies $\delta(s)\equiv 0$. This in turn implies that

$$M_1(s)\equiv\frac{\gamma(s)-p}{R}.$$

Thus, if

$$M_1(0)=\frac{\gamma(0)-p}{R}\qquad\text{then}\qquad M_1(s)=\frac{\gamma(s)-p}{R},\quad\forall s.$$

and M_1 can be seen to be always the outward normal of the sphere, as shown in Figure A.2.

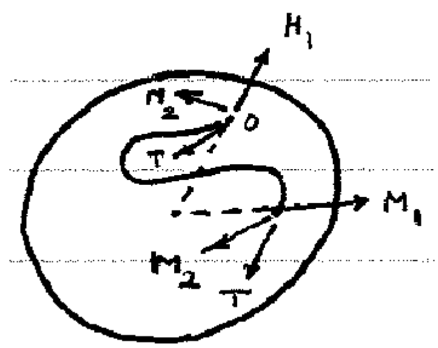


Figure A.2: Natural Frenet frame for a curve constrained to a sphere.

Supplement B

Some Computations Pertaining to Index

1. Signed area.

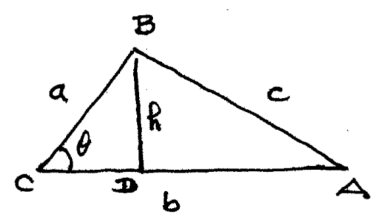


Figure B.1: Triangle for finding area.

The area of a triangle with sides a, b, c is given by adding the areas of two right triangles, BCD and BAD, as shown in Figure B.1.

$$\begin{aligned}\text{Area} &= \frac{1}{2}h \cdot CD + \frac{1}{2}h \cdot AD \\ &= \frac{1}{2}h \cdot b \\ &= \frac{1}{2}ab\sin\theta\end{aligned}$$

Recall from the definition of the vector product, that

$$\begin{aligned}|\vec{CA} \times \vec{CB}| &= |\vec{CA}||\vec{CB}|\sin\theta \\ &= ba\sin\theta\end{aligned}$$

Thus, the oriented/signed area of the triangle ABC is

$$\frac{1}{2}\vec{CA} \times \vec{CB}.$$

2. Area enclosed by a parametrized closed curve.
   Define a parameterized, closed curve γ such that

$$\begin{aligned}\gamma : [0, T] &\to \mathbb{R}^2 \\ t &\mapsto \gamma(t),\end{aligned}$$

with $\gamma(0) = \gamma(T) = P$. As shown in Figure B.2, the enclosed area is obtained by adding up areas of triangles bounded by the vectors r, Δr, and $r + \Delta r$ and taking the limit as $\Delta r \to 0$.

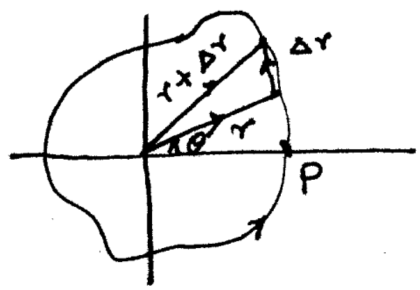


Figure B.2: Triangles within a parametrized closed curve.

Then, we have that

$$\begin{aligned}\text{Area} &\approx \sum_{i=1}^{N} \frac{1}{2} r_i \times (r_i + \Delta r_i) \\ &= \sum_{i=1}^{N} \frac{1}{2} r_i \times \Delta r_i \qquad \text{(since for any } v,\ v \times v = 0).\end{aligned}$$

Taking the limit as the number of terms in the sum goes to ∞, we get,

$$\begin{aligned}\text{Area} &= \oint \frac{1}{2} r(t) \times \frac{dr(t)}{dt} dt \\ &= \int_0^T \frac{1}{2} r(t) \times \dot{r}(t) dt\end{aligned}$$

Let $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. Then,

$$r(t) \times dr(t) = (x\hat{\imath} + y\hat{\jmath}) \times (dx\hat{\imath} + dy\hat{\jmath}) \tag{B.1}$$

$$= (xdy - ydx)\hat{k}, \tag{B.2}$$

where $\hat{k} = \hat{\imath} \times \hat{\jmath}$ is the unit normal to the plane spanned by orthonormal basis vectors $\hat{\imath}, \hat{\jmath}$. Thus,

$$\text{Area} = \frac{1}{2} \oint (xdy - ydx)\hat{k} \tag{B.3}$$

3. Signed area of a curve in polar coordinates.
   Let $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$\begin{aligned}\frac{1}{2}(xdy - ydx) &= \frac{1}{2}\left(r\cos\theta(r\cos\theta d\theta + dr\sin\theta)\right. \\ &\qquad \left.-r\sin\theta(-r\sin\theta d\theta + dr\cos\theta)\right) \\ &= \frac{1}{2}r^2 d\theta\end{aligned}$$

   The signed area enclosed by the curve γ is then given by

$$\text{Area} = \oint \frac{1}{2}r^2 d\theta\hat{k}, \tag{B.4}$$

   with the curved traversed in the counterclockwise manner. If we specialize a γ curve that is the unit circle, centered at the origin,

$$\text{Area} = \frac{1}{2}R^2 \int_0^{2\pi} d\theta\hat{k} \tag{B.5}$$

$$= \pi\hat{k} \tag{B.6}$$

4. Index.

$$\begin{aligned}d\theta &= d\arctan\left(\frac{y}{x}\right) \\ &= \frac{1}{1+\left(\frac{y}{x}\right)^2} d\left(\frac{y}{x}\right) \\ &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}dy - \frac{y}{x^2}dx\right) \\ &= \frac{xdy - ydx}{x^2+y^2}\end{aligned}$$

   Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vectorfield (i.e. the components f_1 and f_2 have continuous first partial derivatives).

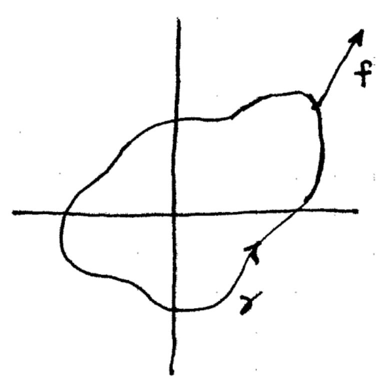


Figure B.3: Vector f at a point (x, y) on a closed curve.

At any point (x, y) in the plane,

$$\theta_f(x, y) = \arctan\left(\frac{f_2(x, y)}{f_1(x, y)}\right), \tag{B.7}$$

if well-defined.

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be a closed curve, not passing through an equilibrium point of $\dot{x} = f(x)$, i.e. a point such that $f(x) = 0$. We let $\hat{f} \triangleq \frac{f}{\|f\|}$ on γ. Then, we have a map

$$\begin{aligned} \tilde{\gamma} : S' &\to S' \\ t &\mapsto \arctan\left(\frac{f_2}{f_1}\right) = \theta_f\Big|_{\gamma(t)}. \end{aligned}$$

Here S' denotes the circle obtained by identifying 0 and 2π.

Then, the index is given by

$$\begin{aligned} \mathrm{I}_\gamma^f &= \frac{1}{2\pi}\oint d\theta_f \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d\theta_f}{dt}\right) dt \end{aligned}$$

The index counts how many times $\tilde{\gamma}$ winds around the circle.

5. Index I_γ^f for γ, a closed orbit of a vector field.
Everywhere on a closed orbit we have,

$$f = \frac{d\gamma}{dt},$$

with $f \neq 0$ on γ.

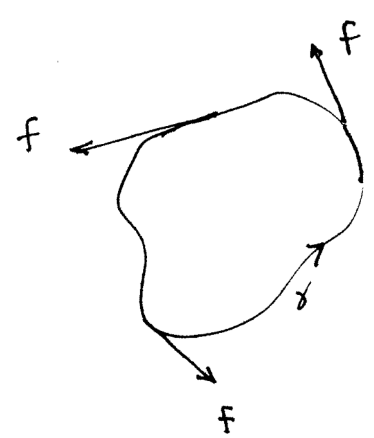


Figure B.4: Vector field evaluated on a closed orbit.

We may write

$$\begin{aligned}\frac{d\theta_f}{dt} &= \frac{d}{dt}\arctan\left(\frac{f_2}{f_1}\right)\\ &= \frac{d}{dt}\arctan\left(\frac{\dot\gamma_2}{\dot\gamma_1}\right)\\ &= \frac{\dot\gamma_1\ddot\gamma_2 - \dot\gamma_2\ddot\gamma_1}{\dot\gamma_1^2+\dot\gamma_2^2}.\end{aligned}$$

The integral

$$\mathrm{I}_\gamma^f = \frac{1}{2\pi}\int_0^{2\pi}\frac{\dot\gamma_1\ddot\gamma_2 - \dot\gamma_2\ddot\gamma_1}{\dot\gamma_1^2+\dot\gamma_2^2}dt \tag{B.8}$$

can be effectively computed by a change of variable (parametrization). Let

$$\begin{aligned}s(t) &= \int_0^t \|\dot\gamma(t)\|\,dt\\ &= \int_0^t (\dot\gamma_1^2+\dot\gamma_2^2)^{1/2}dt\end{aligned}$$

denote the length of the arc $\{\gamma(\sigma) : 0 \le \sigma \le t\}$. The total length of the closed curve is therefore $s(2\pi)$. By hypothesis that $\dot\gamma \neq 0$ on γ, $t \mapsto s(t)$ is a strict monotone increasing function. Hence, it can be inverted to obtain $s \mapsto s(t)$. We can think of γ as parametrized by arc length by substitution, $s \mapsto \gamma(t(s))$. Denote by ($'$) the derivative operator $\frac{d}{ds}$ and let $\nu(t) = \sqrt{\dot\gamma_1^2+\dot\gamma_2^2}$ be the speed.

Then,

$$\begin{aligned}\frac{d}{dt} &= \frac{d}{ds}\frac{ds}{dt}\\ &= \nu\frac{d}{ds}.\end{aligned}$$

Hence, $\dot\gamma = \nu\gamma'$, which implies that

$$\begin{aligned}\dot\gamma_1\ddot\gamma_2 - \dot\gamma_2\ddot\gamma_1 &= \nu\gamma_1'(\gamma_2''\nu^2+\gamma_2'\dot\nu) - \nu\gamma_2'(\gamma_1''\nu^2+\gamma_1'\dot\nu)\\ &= \nu^3(\gamma_1'\gamma_2''-\gamma_2'\gamma_1''),\end{aligned}$$

and we can substitute this into the index formula to yield,

$$\begin{aligned}
\mathrm{I}_\gamma^f &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\nu^3}{\nu^2}(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')\nu dt \\
&= \frac{1}{2\pi}\oint (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')ds \\
&= \frac{1}{2\pi}\oint (\gamma_1'\hat{i} + \gamma_2'\hat{j}) \times \frac{d}{ds}(\gamma_1'\hat{i} + \gamma_2'\hat{j})ds \qquad \text{(Recall signed area)} \\
&= \frac{1}{2\pi}2(\text{area enclosed by curve } s \mapsto \gamma'(s))
\end{aligned}$$

But, we also have

$$\begin{aligned}
\|\gamma'\| &= \left\|\dot{\gamma}\frac{1}{\nu}\right\| \\
&= \frac{\|\dot{\gamma}\|}{\nu} \\
&= \frac{\nu}{\nu} = 1
\end{aligned}$$

i.e. the curve $s \mapsto \gamma'(s)$ is the unit circle. Hence, it encloses area π, and hence

$$\begin{aligned}
\mathrm{I}_\gamma^f &= \frac{1}{2\pi} \cdot 2 \cdot \pi \\
&= +1.
\end{aligned}$$

We have shown that the index of a vector field with respect to γ, a closed orbit of f, is $+1$.

Supplement C

Proof of a Technical Lemma

We now consider the proof of a technical lemma used in the main theorem for stability of time-varying systems.

Lemma C.1

Let

$$\dot{y} = -\alpha(y) \qquad y(t_0) = y_0,$$

where $\alpha(\cdot)$ is a class $\mathcal{K}$ function. Assume further that $\alpha(\cdot)$ is locally Lipschitz. Suppose $\alpha(\cdot)$ is defined on $[0, a)$. Then, for all $0 \le y_0 \le a$, the dynamics have a unique solution $y(t)$ defined $\forall t \ge t_0$. Moreover,

$$y(t) = \sigma(y_0, t - t_0),$$

where $\sigma(\cdot, \cdot)$ is a class $\mathcal{KL}$ function on $[0, a) \times [0, \infty)$. $\square$

Proof of Lemma C.1

$\alpha(\cdot)$ is locally Lipschitz implies there exists a unique solution $\forall y_0 \ge 0$. Since $\dot{y}(t) < 0$ whenever $y(t) > 0$, the solution $y(t) \le y_0$, $\forall t \ge t_0$. Therefore, the solution is bounded and can be extended $\forall t \ge t_0$.

By integration,

$$\eta_{y_0}(y) \triangleq -\int_{y_0}^{y} \frac{dx}{\alpha(x)} = t - t_0$$

gives the sojourn time map (i.e. how long it takes to get to y from y_0) defined on $(0, y_0)$.

Let $\eta(y) \triangleq \eta_b(y)$ for $0 < b < a$. $\eta(\cdot)$ is strictly decreasing, differentiable on $(0, a)$. Moreover, $\eta(y) \to \infty$ as $y \to 0$. To see this, note that $y(t) \to 0$ as $t \to \infty$, since $\dot{y}(t) < 0$, for $y(t) > 0$. This can only happen asymptotically as

$t \to \infty$ (i.e. it cannot happen in finite time without violating uniqueness).

Notice that since $b < a$,

$$\eta(a) = -\int_b^a \frac{dx}{\alpha(x)} = -c$$

for some $c > 0$.

We have $\eta : (0, a) \to (-c, \infty)$ and $\eta^{-1} : (-c, \infty) \to (0, a)$ is also well-defined, since η is a strictly decreasing function of its argument.

Since

$$(t-t_0) = \eta_{y_0}(y) = -\int_{y_0}^y \frac{dx}{\alpha(x)} = -\left(\int_{y_0}^b \frac{dx}{\alpha(x)} + \int_b^y \frac{dx}{\alpha(x)}\right) = -\eta(y_0)+\eta(y),$$

we can write

$$\eta(y) = \eta(y_0) + (t - t_0).$$

Then for $y_0 > 0$,

$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0)$$

and

$$y(t) \equiv 0 \qquad \textit{if} \quad y_0 = 0.$$

Now define

$$\sigma(r, s) = \begin{cases} \eta^{-1}(\eta(r) + s) & r > 0 \\ 0 & r = 0 \end{cases}$$

Then,

$$y(t) = \sigma(y_0, t - t_0),$$

$\forall t \geq t_0$, $y_0 \geq 0$. Note that σ is continuous since both η and η^{-1} are continuous and $\lim_{x\to\infty} \eta^{-1}(x) = 0$.

For fixed s,

$$\begin{aligned} \frac{\partial \sigma(r, s)}{\partial r} &= \frac{\partial}{\partial r}\left(\eta^{-1}(\eta(r) + s)\right) \\ &= \frac{\alpha(\sigma(r, s))}{\alpha(r)} > 0 \qquad (\textit{this formula takes a little bit of calculus}). \end{aligned}$$

is strictly increasing in r.

For fixed r,

$$\frac{\partial \sigma(r, s)}{\partial r} = -\alpha(\sigma(r, s)) < 0 \qquad (\textit{this formula takes a little bit of calculus})$$

is strictly decreasing in s.

Furthermore $\sigma(r, s) \to 0$ as $s \to \infty$ since $\eta^{-1} \to 0$ as its argument $\to \infty$. So we have shown σ is class $\mathcal{KL}$. ∎

Example C.1. (a) $\alpha(y) = -\gamma y, \gamma > 0$.

$$\sigma(r, s) = r \exp(-\gamma s).$$

(b) $\alpha(y) = -ky^2, k > 0$.

$$\begin{aligned}\eta(y) &= -\int_b^y \frac{dx}{kx^2} \\ &= \frac{1}{k}\left(\frac{1}{y} - \frac{1}{b}\right)\end{aligned}$$

and

$$\sigma(r, s) = \frac{1}{\frac{1}{r} + ks}.$$