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ENEE 765 Lecture 6 (Analysis of deterministic identification)

We need to define some function space.
For $1 \leq p < \infty$, let

$$L_p^m := \{ \underset{\text{measurable}}{f : [0,\infty) \to \mathbb{R}^m} \mid \int_0^\infty \|f(t)\|_{\mathbb{R}^m}^p \, dt < \infty \}$$

The integral is in the sense of Lebesgue. $\|\cdot\|_{\mathbb{R}^m}$ denotes ~~any~~ norm on the

Euclidean space $\mathbb{R}^m$ since all such norms are equivalent. [Two norms in $\mathbb{R}^m$ denoted as $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if there exists k_1, k_2, k_3, k_4 all positive such that

$$k_1 \|x\|_a \leq \|x\|_b \leq k_2 \|x\|_a$$

and

$$k_3 \|x\|_b \leq \|x\|_a \leq k_4 \|x\|_b \,]$$

If the function $f \in L_p^m$, then we define the function space norm

$$\|f\|_p = \left(\int_0^\infty \|\cancel{f}\, f(t)\|_{\mathbb{R}^m}^p \, dt \right)^{1/p}.$$

$$L_\infty^m := \left\{ \underset{\text{measurable}}{f : [0,\infty) \to \mathbb{R}^m} \;\middle|\; \begin{array}{l} \|f(t)\|_{\mathbb{R}^m} < M \text{ for some } M > 0 \\ \text{except possibly on} \\ \text{a set of measure zero} \end{array} \right\}$$

essentially bounded

We then define the <u>function space</u> norm to be

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} \|f(t)\|$$

on the space L_∞^m

It is convenient to drop the superscript m on L_p^m and L_∞^m as it will be apparent from the context. Similarly we drop the subscript $\mathbb{R}^m$ from $\| \ \|_{\mathbb{R}^m}$ as it will be apparent what is meant from the context.

The aforementioned normed linear spaces are <u>Complete</u> [i.e. every Cauchy sequence in such a space converges]. They are referred to as <u>Banach spaces</u>.

When $p = 2$, and $m = 1$, we recognize an positive definite inner product on L_2 by setting

$$\langle f, g \rangle = \int_0^\infty f(t) g(t)\, dt$$

and

$$\|f\|_2 = \langle f, f \rangle^{1/2}$$

so L_2 is a Hilbert space over the Reals.

We now prove a set of properties of the gradient algorithm from Lecture 5.

(a) Consider $\dot{\phi} = \dot{\theta} = -\gamma e_1 w \qquad \gamma > 0$

$$e_1 = \phi^T w$$

where $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is assumed to be piecewise continuous. Then

$$e_1 \in L_2 \quad \text{and} \quad \phi \in L_\infty$$

Proof $\quad \dot{\phi} = -\gamma w w^T \phi \qquad (*)$

$$V(\phi) = \frac{1}{2} \phi^T \phi$$

satisfies

$$\begin{aligned} \dot{V}(\phi) &= \frac{d}{dt} V(\phi(t)) \Big|_{(*)} \\ &= \phi^T \dot{\phi} \\ &= -\gamma \phi^T w w^T \phi \\ &= -\gamma (w^T \phi)^2 \\ &\leq 0 \end{aligned}$$

Hence $0 \leq V(\phi(t)) \leq V(\phi(0)) \quad \forall \ t \geq 0$

$\Rightarrow \boxed{\phi \in L_\infty}$

$V(\phi(t))$ is monotone decreasing and is bounded below by 0. Hence $\lim_{t \to \infty} V(\phi(t))$ exists, is finite, and denoted $V_\infty \geq 0$.

But,

$$\int_0^\infty e_1^2(t)\,dt = \int_0^\infty \left(\phi^T(t)\,w(t)\right)^2 dt$$

$$= \int_0^\infty -\frac{\dot{V}(\phi(t))}{\gamma}\,dt$$

$$= \frac{V(\phi(0)) - V_\infty}{\gamma} < \infty$$

$$\Rightarrow \quad \boxed{e_1 \in L_2}\ . \qquad \square$$

(b) As a variant on the gradient algorithm consider the normalized version,

$$\dot{\phi} = \dot{\theta} = \frac{-\gamma\, e_1 w}{1 + \varepsilon_0 w^T w}\ ; \quad e_1 = \phi^T w$$

with $\varepsilon_0 > 0$, $\gamma > 0$. Assume $w : \mathbb{R}_+ \to \mathbb{R}^{2n}$ is piecewise continuous. ~~and~~ Then

(i) $\dfrac{e_1}{\sqrt{1 + \varepsilon_0 w^T w}} \in L_2 \cap L_\infty$

(ii) $\phi \in L_\infty$, $\dot{\phi} \in L_2 \cap L_\infty$

(iii) $\beta = \dfrac{\phi^T w}{1 + \|w_t\|_\infty} \in L_2 \cap L_\infty$

(where, $\|w_t\|_\infty = \max\limits_{1 \le i \le 2n} |w_i(t)|$

PROOF Let $V(\phi) = \phi^T \phi$

Then $\dot{V} = 2\phi^T \dot{\phi} = 2\phi^T \left(\dfrac{-\gamma\, e_1 w}{1 + \varepsilon_0 w^T w}\right)$

$$= \frac{-2\gamma\ \phi^T W W^T \phi}{1 + \varepsilon_0 W^T W} = \frac{-2\gamma\ e_1^2}{1 + \varepsilon_0 W^T W} \leq 0$$

Then $0 \leq V(\phi(t)) \leq V(\phi(0)) \quad \forall\ t \geq 0$

Hence $V(\phi) \in L_\infty$

$$\|\phi(t)\|_2 = \sqrt{\sum_{i=1}^{2n} (\phi_i(t))^2}$$

$$\leq \sqrt{V(\phi(0))} \qquad \forall\ t \geq 0$$

$$\Rightarrow \boxed{\phi \in L_\infty} \quad \text{(ii)}$$

$$\left| \frac{e_1(t)}{\sqrt{1+\varepsilon_0 W^T(t) W(t)}} \right| = \left| \frac{\phi^T(t)\, W(t)}{\sqrt{1+\varepsilon_0 W^T(t) W(t)}} \right|$$

$$= \frac{|\phi^T(t)\, W(t)|}{\sqrt{1+\varepsilon_0 W^T(t) W(t)}}$$

$$\leq \frac{\|\phi(t)\|_2\ \|W(t)\|_2}{\sqrt{1+\varepsilon_0 W^T(t) W(t)}} \quad \text{(Cauchy-Schwarz)}$$

$$\leq \sqrt{V(\phi(0))} \cdot \frac{1}{\sqrt{\varepsilon_0}} \sqrt{\frac{W^T(t) W(t)}{\frac{1}{\varepsilon_0} + W^T(t) W(t)}}$$

$$\leq \frac{1}{\sqrt{\varepsilon_0}} \sqrt{V(\phi(0))}$$

$$\Rightarrow \boxed{\frac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_\infty}$$

$$|\beta(t)| = \frac{|\phi^T(t)\, w(t)|}{1 + \max\limits_{1 \le i \le 2n} |w_i(t)|}$$

$$\le \sum_{j=1}^{2n} |\phi_j(t)| \frac{|w_j(t)|}{1 + \max\limits_{1 \le i \le 2n} |w_i(t)|}$$

$$\le \sum_{j=1}^{2n} |\phi_j(t)|$$

$$\le 2n \, \|\phi(t)\|_2$$

$$\le 2n \sqrt{V(\phi(0))}$$

$$\Rightarrow \quad \boxed{\beta \in L_\infty} \checkmark$$

Thus we have shown that V, ϕ, $\frac{e_1}{\sqrt{1+\varepsilon_0 w^T w}}$, β all belong to L_∞.
We now need to show the "L_2 part".

$$\dot{\phi} = -\frac{\gamma\, w\, e_1}{1 + \varepsilon_0 w^T w}$$

$$= -\frac{\gamma}{\varepsilon_0} \left(\frac{\varepsilon_0\, w\, w^T \phi}{1 + \varepsilon_0 w^T w} \right)$$

<u>aside:</u> for matrix A

$$\|Ax\|_2 = \sqrt{x^T A^T A x}$$

$$\le \sqrt{\lambda_{max}(A^T A)} \, \sqrt{x^T x}$$

$$= \sqrt{\lambda_{max}(A^T A)} \, \|x\|_2$$

Using the above Aside,

let $A = W W^T$.

$$\lambda_{max}(A^T A) = \lambda_{max}(W W^T W W^T)$$
$$= \lambda_{max}((W^T W) W W^T)$$
$$= (W^T W)^2$$

Hence

$$\| W W^T \phi \|_2 \le \sqrt{\lambda_{max}((W W^T)^T (W W^T))} \; \| \phi \|_2$$
$$= \sqrt{(W^T W)^2} \; \| \phi \|_2$$
$$= W^T W \; \| \phi \|_2 \qquad \text{(all evaluated at } t\text{)}$$

$$\Rightarrow \| \dot{\phi}(t) \|_2 \le \frac{\gamma}{\varepsilon_0} \frac{(\varepsilon_0 W^T(t) W(t))}{(1 + \varepsilon_0 W^T(t) W(t))} \| \phi(t) \|_2$$
$$\le \frac{\gamma}{\varepsilon_0} \| \phi(t) \|_2$$
$$\le \frac{\gamma}{\varepsilon_0} \sqrt{V(\phi(0))}$$

$$\Rightarrow \boxed{\dot{\phi} \in L_\infty} \checkmark$$

$$\int_0^\infty \left[\frac{e_1(t)}{\sqrt{1 + \varepsilon_0 W^T(t) W(t)}} \right]^2 dt = \int_0^\infty \frac{e_1^2(t)}{1 + \varepsilon_0 W^T(t) W(t)} dt$$

$$= \int_0^\infty - \frac{\dot{V}(\phi(t))}{2\gamma} dt = \frac{V(\phi(0)) - V_\infty}{2\gamma}$$ exist and is finite

Thus $\dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_2$

We have thus far shown

$$\boxed{\dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_2 \cap L_\infty \qquad (i)}$$

$$\beta(t) = \frac{\phi^T(t)\, W(t)}{1 + \max\limits_{1 \le i \le 2n} |W_i(t)|}$$

$$= \frac{\phi^T(t)\, W(t)}{\sqrt{1+\varepsilon_0 W^T(t)\, W(t)}} \cdot \frac{\sqrt{1+\varepsilon_0 W^T(t) W(t)}}{1 + \max\limits_{1 \le i \le 2n} |W_i(t)|}$$

This belongs to L_2 (See above)

if we show this belongs to L_∞ then we can conclude $\beta \in L_2$

Recall, in finite dimensions, say in $\mathbb{R}^k$

$$\|x\|_\infty = \max_{1 \le i \le k} |x_i| \le \|x\|_2 \le \sqrt{k}\, \|x\|_\infty$$

Hence

$$\frac{\sqrt{1+\varepsilon_0 \|w(t)\|_2^2}}{1 + \max\limits_{1 \le i \le 2n} |w_i(t)|} \le \frac{\sqrt{1+\varepsilon_0 \|w(t)\|_2^2}}{1 + \frac{1}{\sqrt{2n}} \|w(t)\|_2}$$

The right hand side of the above inequality has the form, $\sqrt{f(y)}$, where with $y \geq 0$

$$f(y) = \frac{1+\varepsilon_0 y^2}{\left(1 + \frac{1}{\sqrt{2n}} y\right)^2}$$

$$= \frac{1+\varepsilon_0 y^2}{1 + \frac{1}{2n} y^2 + \sqrt{\frac{2}{n}}\, y}$$

$$\to 2n\varepsilon_0 \quad \text{as } y \to \infty.$$

Pick a $\delta > 0$. Then $\exists\, Y > 0$ such that, $\forall y > Y$, $f(y) < 2n\varepsilon_0 + \delta$. On the other hand, on the closed and bounded set $[0, Y]$, $f(y)$ has a maximum f_{max} due to continuity of $f(y)$ – Weierstrass Theorem.

Hence $f(y) < \max(f_{max}, 2n\varepsilon_0 + \delta)$

Thus $\sqrt{f} \in L_\infty$. Thus we have shown $\beta \in L_2$

We have now shown $\beta \in L_2 \cap L_\infty$ (iii) ✓

$$\|\dot{\phi}(t)\|_2^2 = \frac{\gamma^2 e_1^2(t)\, w^T(t) w(t)}{(1+\varepsilon_0 w^T(t) w(t))^2}$$

$$= \frac{\gamma^2}{\varepsilon_0} \frac{e_1^2(t)}{(1+\varepsilon_0 w^T w(t))} \frac{\varepsilon_0 w^T(t) w(t)}{(1+\varepsilon_0 w^T(t) w(t))}$$

$$\leq \frac{\gamma^2}{\varepsilon_0} \frac{e_1^2(t)}{(1+\varepsilon_0 w^T(t) w(t))}$$

We have already shown $\frac{e_1}{\sqrt{1+\varepsilon_0 w^T w}} \in L_2$

Hence $\dot{\phi} \in L_2$ which then $\Rightarrow$ $\boxed{\dot{\phi} \in L_2 \cap L_\infty \text{ (ii)}}$

$\square$

(c) Suppose $\dot{\phi} = \dot{\theta} = -\gamma e_1 w$

$$e_1 = \phi^T w + \varepsilon$$

(accounting for observer gap/error)

where $\varepsilon(t) \to 0$ exponentially as $t \to \infty$

Then $e_1 \in L_2$, $\phi \in L_\infty$

PROOF: Define $V(\phi) \triangleq \phi^T \phi + \frac{\gamma}{2} \int_t^\infty \varepsilon^2(\sigma)\, d\sigma$

$$\dot{V} = 2\phi^T \dot{\phi} - \frac{\gamma}{2} \varepsilon^2(t)$$

$$= 2\phi^T(-\gamma(\phi^T w + \varepsilon) w) - \frac{\gamma}{2}\varepsilon^2$$

$$= -2\gamma(\phi^T w)^2 - 2\gamma(\phi^T w)\varepsilon - \frac{\gamma}{2}\varepsilon^2$$

$$= -2\gamma\left(\phi^T w + \frac{\varepsilon}{2}\right)^2 \leq 0$$

$0 \le V(\phi(t))$ and $V(\phi(t))$ is monotone decreasing with increasing t $\Rightarrow$ $\lim_{t \to \infty} V(\phi(t))$ exists,

is finite $= V_\infty \ge 0$

$$\|\phi(t)\|_2 \le \sqrt{V(\phi(0))}$$

$$= \left(\|\phi(0)\|_2^2 + \int_0^\infty \varepsilon^2(\sigma)\, d\sigma \right)^{1/2}$$

$$\Rightarrow \boxed{\phi \in L_\infty}$$

$$\int_0^\infty \left(\phi^T(t) w(t) + \frac{\varepsilon}{2}(t)\right)^2 dt$$

$$= \int_0^\infty - \frac{\dot{V}(\phi(t))}{2\gamma}\, dt$$

$$= \frac{V(\phi(0)) - V_\infty}{2\gamma}$$

$$< \infty$$

Thus $\phi^T w + \frac{\varepsilon}{2} \in L_2$

On the other hand $\frac{\varepsilon}{2} \in L_2$ (since by hypothesis $\varepsilon(t)$ decays exponentially to 0 as $t \to \infty$).

Hence $\boxed{e_1 = (\phi^T w + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2} \in L_2}$

(recall L_2 is a vector space). $\square$

Some recapitulation

In proving the properties listed under (a) (b) (c) above, we have used

(1) $w(\cdot)$ is piecewise continuous $\Rightarrow$ differential equation $\dot{\phi} = -\gamma w w^T \phi$ has well-defined (series) solution.

(2) A monotone decreasing function $g : [0, \infty) \to \mathbb{R}$ which satisfies a lower bound $g(t) \geq c$ for c a constant, has a limit

(3) Cauchy - Schwartz inequality

(4) for a symmetric matrix Q of size $k \times k$,

$$\lambda_{min}(Q)\, x^T x \leq x^T Q x \leq \lambda_{max}(Q)\, x^T x$$

$\forall\ x \in \mathbb{R}^n$.

[the ration $\dfrac{x^T Q x}{x^T x}$ is called

Rayleigh quotient]

(5) If $f \in L_2$, $g \in L_\infty$ with $\|g\| = M$ then

$$f \cdot g \in L_2$$

Since $\displaystyle\int_0^\infty ((f \cdot g)(t))^2 dt \leq \int_0^\infty f^2(t) \cdot M^2 dt \leq M^2 \int_0^2 f^2(t)\, dt$

In the proof of parts of properties (b), we follow the sequence of proofs

(ii) $\phi \in L_\infty$

(i) $\dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_\infty$ $\qquad \dfrac{e_1}{\sqrt{1+\varepsilon_0 W^T W}} \in L_2$

(iii) $\beta \in L_\infty$ $\qquad \beta \in L_2$

(ii) $\dot\phi \in L_\infty$ $\qquad \dot\phi \in L_2$

In what follows we seek state a stability theorem for our deterministic identifier(s) under additional hypotheses. We need a technical result of wide applicability in control theory known as Barbalat's Lemma (L. Barbalat, "Systèmes d'équations différentielles d'oscillations non linéaires", Rev. Math. Pures Appl. 4 267–270 (1959)), and a corollary to it.

> A good lemma is worth a thousand theorems
> — Doron Zeilberger
> Rutgers University

Barbalat's Lemma

Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$. Suppose that

$$\lim_{t \to \infty} \int_0^t f(\sigma)\, d\sigma$$

exists and is finite. Then

$$f(t) \to 0 \quad \text{as } t \to \infty .$$

Proof: We prove by contradiction.

Suppose there is a constant $k_1 > 0$ s.t. for every $T > 0$, we can find $T_1 \geq T$ such that $|f(T_1)| \geq k_1$. Since f is uniformly continuous, there is a positive constant k_2 s.t.

$$|f(t+\tau) - f(t)| < \frac{k_1}{2} \quad \forall\, t \geq 0$$

and all $0 \leq \tau \leq k_2$.

Hence,

$$\begin{aligned} |f(t)| &= |f(t) - f(T_1) + f(T_1)| \\ &\geq |f(T_1)| - |f(t) - f(T_1)| \\ &> k_1 - \frac{k_1}{2} = \frac{k_1}{2} \\ &\qquad \forall\, t \in [T_1, T_1 + k_2] \end{aligned}$$

Therefore,

$$\left| \int_{T_1}^{T_1+k_2} f(t)\,dt \right| = \int_{T_1}^{T_1+k_2} |f(t)|\,dt$$

$$> \frac{1}{2} k_1 k_2$$

where the equality holds since $f(t)$ has same sign $\forall\ t \in [T_1, T_1+k_2]$.

Thus $\int_0^t f(\sigma)\,d\sigma$ cannot converge to finite limit as $t \to \infty$, a contradiction.

This completes the proof $\square$

Corollary

Let $g, \dot{g} \in L_\infty$ and $g \in L_p$ for $1 \le p < \infty$. Then $g(t) \to 0$ as $t \to \infty$

Proof Let $f(t) = |g(t)|^p$. Then $g, \dot{g}$ bounded $\Rightarrow$ f is uniformly continuous.

$g \in L_p \Rightarrow \lim_{t\to\infty} \int_0^t f(\sigma)\,d\sigma$ exists and is finite and $= (\|g\|_p)^p$

By Barbalat's Lemma $f(t) \to$ as $t \to \infty$

Equivalently $g(t) \to 0$ as $t \to \infty$ $\square$

Convergence / Stability Theorem for Identifier,

Consider the deterministic identification problem for a linear, time-invariant, finite-dimensional, strictly proper plant subject to a reference input signal $r(\cdot)$ which is piecewise continuous and bounded.

Suppose that the plant is stable (all poles in the open left half-plane $\mathbb{C}^-$) or placed in a known linear feedback loop such that r and y_p are bounded. Then, for the gradient algorithm with or without normalization, or the least squares algorithm with normalization and re-setting, the following hold:

(i) $e_1 \in L_2 \cap L_\infty$

(ii) $e_1 \to 0$ as $t \to \infty$

(iii) $\phi, \dot{\phi} \in L_\infty$

(iv) $\dot{\phi} \in L_2 \cap L_\infty$

and

$$\dot{\phi} \to 0 \quad \text{as} \quad t \to \infty.$$

Proof In properties (a) (b)(c) <u>above</u> we have shown for the gradient and normalized gradient algorithms,

$$e_1 \in L_2 \qquad \phi \in L_\infty$$

r and y_p bounded $\Rightarrow$ w and $\dot{w}$ bounded

$\phi, \dot{\phi} \in L_\infty \Rightarrow e_1 = \phi^T w$ and
$\dot{e}_1 = \dot{\phi}^T w + \phi^T \dot{w} \in L_\infty$

$e_1 \in L_2$ and $\dot{e}_1, e_1 \in L_\infty$

$$\Rightarrow e_1(t) \to 0 \text{ as } t \to \infty$$

(by Barbalat's Lemma and its corollary above).

Similarly

$$\dot{\phi} \in L_2 \text{ and } \phi, \dot{\phi} \in L_\infty$$

$$\Rightarrow \dot{\phi}(t) \to 0 \text{ as } t \to \infty$$

(Here we need to appeal to Barbalat's lemma with functions taking values in $2n$ dimensional vector spaces ~ an easy generalization) $\square$