Lecture 1a

Modeling an individual agent / unit in a collective begins with an abstraction.

For starters, consider the ~~particle~~ agent modeled as

$$\boxed{f = ma = m\dot{v} = m\ddot{r}}$$

representing trajectory $t \longmapsto r(t)$ generated by (integrating) the force law $t \mapsto f(t)$.

This calculation is easy (variation of constants formula from linear system theory).

But the results are sensitive to the choice of "reference frame in the laboratory". Is there a way to re-express this agent model so as to be indifferent to the laboratory frame?

We construct such a representation.

Let $t \longmapsto r(t) \quad 0 \le t \le T$ be a curve in $\mathbb{R}^3$, starting at $r_0 = r(0)$.

As we shall see below we need $r(t)$ to be C^3 i.e thrice continuously differentiable.

Let $v = \| \frac{dr}{dt} \| = \langle \frac{dr}{dt}, \frac{dr}{dt} \rangle^{1/2} = \left(\frac{dr}{dt} \cdot \frac{dr}{dt} \right)^{1/2} \geq 0$

denote the speed in terms of Euclidean inner product (dot product).

Then

$$s(t) = \int_0^t \| \frac{dr}{d\sigma} \| \, d\sigma$$

= length of the curve from $r(0)$ to $r(t)$.

Clearly $s(t) \geq s(t')$ if $t \geq t'$ (<u>monotone increasing</u>)

<u>Hypothesis 1</u>. $\dot{r}(t) \neq 0$ for $0 \leq t \leq T$ (<u>regularity</u>)

Then $s(t) > s(\sigma)$ if $t > \sigma$ (strict monotonicity), and <u>in principle</u> we can <u>invert</u> the map $t \mapsto s(t)$ to obtain $t = t(s)$, and hence be able to express the curve in terms of the <u>arc-length parameter</u> s.

Formally we write,

$$s \longmapsto r(s) = r(t(s)).$$

The tangent vector to the curve r at s is

$$T(s) \triangleq \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{1}{v}\frac{dr}{dt}$$

Note: $(\ )' \triangleq \frac{d}{ds} = \frac{1}{\nu}\frac{d}{dt}$

Hence,

$$\|T(s)\| = \left\|\frac{dr}{dt}\right\| \frac{1}{\nu} = 1 \quad \forall\ t \geq 0 .$$

Thus, in the arc-length parametrization the curve has unit speed; and we call T the unit tangent vector.

Change of Laboratory Frame by translation b and rotation P produces curve representation

$$\tilde{r}(t) = P\, r(t) + b$$

which however does not alter speed and hence leaves invariant the arclength $s(t)$, upto time t.

Differentiate both sides of $T(s)\cdot T(s) \equiv 1$ to obtain $T'(s)\cdot T(s) \equiv 0$.

Define classical curvature $\boxed{\kappa(s) = \left\|\frac{dT}{ds}\right\|}$ $= \|r''(s)\| \geq 0$ at s. This is also clearly invariant to translation and rotation.

Denote $S = s(T)$ = length of curve over $0 \leq t \leq T$.

Proposition [Exercise 1]

$\kappa(s) \equiv 0$ on an interval $0 \le s \le S$ iff the curve $s \mapsto \gamma(s)$ is a straight line over the same interval ∎

If for some interval $(s_1-\varepsilon, s_1+\varepsilon)$, $\varepsilon > 0$ about $s = s_1$, there is one point in the interval at which curvature $\kappa > 0$, then (by continuity of $\kappa(s)$) it is true at every point on this interval, provided ε is small ~~enough~~ enough. In that case, on this curve fragment one can define an orthonormal frame (Frenet-Serret frame)

$$\{T(s), N(s), B(s)\}$$

where $N(s) = T'(s)/\kappa(s)$ is a unit vector orthogonal to T, and hence called the Normal, and $B(s) = T(s) \times N(s)$ is also a unit vector, orthogonal to T and N and hence called the bi-Normal.

The frame is said to be co-moving and is adapted in the sense that $T(s)$ is tangent at s.

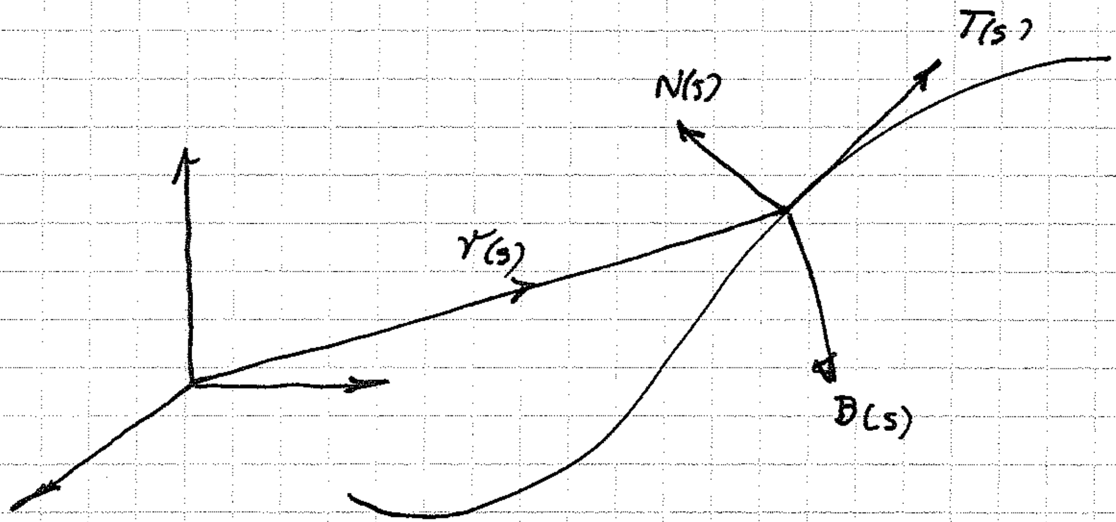

Figure 1. Moving Frame (Frenet-Serret)

From here on we assume,

Hypothesis 2 From here on we assume

$$\kappa(s) > 0 \qquad \forall \; s \in [0, S]$$

(non-degeneracy).

Under regularity and nondegeneracy one can derive a set of differential equations for the triad $\{T(s), N(s), B(s)\}$.

In general, for any orthonormal triad $\{F_1(s), F_2(s), F_3(s)\}$ of unit vectors in $\mathbb{R}^3$,

$$F_i(s) \cdot F_j(s) \equiv \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Differentiating both sides

$$F_i' \cdot F_j + F_i \cdot F_j' \equiv 0.$$

Let us uniquely represent

$$F_j'(s) = \sum_{k=1}^{3} a_j^k(s)\, F_k(s), \qquad j = 1, 2, 3$$

in terms of the matrix $[a_j^k]$.

Then

$$\left(\sum_{k=1}^{3} a_i^k(s)\, F_k(s)\right) \cdot F_j(s) + F_i(s) \cdot \left(\sum_{k=1}^{3} a_j^k(s)\, F_k(s)\right)$$

$$\equiv 0.$$

Equivalently

$$a_i^j(s) + a_j^i(s) \equiv 0,$$

or in other words, the matrix $[a_j^k]$ is skew-symmetric. We specialize now to the case when $F_1 \triangleq T$; $F_2 \triangleq N$; $F_3 \triangleq B$.

$$F_1' = T' = \sum_1^3 a_1^k F_k = \kappa N = \kappa T_2$$

Thus $a_1^2 = \kappa = -a_2^1$; $a_1^3 = 0 = -a_3^1$

Let us call $a_2^3 = \tau = -a_3^2$.

Then

$$N' = F_2' = \sum_{k=1}^{3} a_2^k F_k = -\kappa T + \tau B$$

and

$$B' = F_3' = \sum_{1}^{3} a_3^k F_k = -\tau N$$

From the last equation, we see the <u>interpretation</u>

$$\boxed{\tau = - N \cdot B'} \qquad (\underline{\text{torsion}})$$

Taken together

$$\begin{aligned} T' &= \quad \kappa N \\ N' &= -\kappa T \quad + \tau B \\ B' &= \quad -\tau N \end{aligned}$$

or in matrix differential equation form

$$\frac{d}{ds}\begin{bmatrix} T & N & B \end{bmatrix} = \begin{bmatrix} T & N & B \end{bmatrix} \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

We now have written a <u>generative model</u>

for the curve $s \longmapsto \gamma(s)$

$$\left[\begin{array}{ccc|c} T & N & B & \gamma \\ \hline 0 & 0 & 0 & 1 \end{array}\right]' = \left[\begin{array}{ccc|c} T & N & B & \gamma \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & -\kappa & 0 & 1 \\ \kappa & 0 & -\tau & 0 \\ 0 & \tau & 0 & 0 \\ \hline & & & 0 \end{array}\right].$$

It is generative in the sense that given a <u>program</u> of curvature $\kappa(s)$ and torsion $\tau(s)$ over $s \in [0, S]$, one simply needs to integrate the above matrix differential equations (<u>Frenet - Serret equations</u>) to produce the curve $s \mapsto \gamma(s)$.

<u>Proposition</u> (Exercise 2)

A curve $s \mapsto \gamma(s)$ is <u>planar</u> iff $\tau(s) \equiv 0$.

(Hint: recall that a curve $s \mapsto \gamma(s)$ is said to be planar if there is <u>fixed</u> <u>nonzero</u> vector μ such that

$$\mu \cdot \gamma(s) \equiv \text{constant}$$

Think of μ as a normal to the plane.)

As a final step, we can revert back to the original parametrization t, and express the generative model as

$$\frac{d}{dt}\left[\begin{array}{ccc|c} T & N & B & r \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \nu \left[\begin{array}{ccc|c} T & N & B & r \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & -\kappa & 0 & 1 \\ \kappa & 0 & -\tau & 0 \\ 0 & \tau & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

Exercise 3

For a regular, nondegenerate curve r show that

$$\kappa = \frac{\| \dot{r} \times \ddot{r} \|}{\| \dot{r} \|^3}$$

and

$$\tau = \frac{\dot{r} \cdot (\ddot{r} \times \dddot{r})}{\| \dot{r} \times \ddot{r} \|^2}$$

Hint: (i) Use the BAC−CAB rule from vector analysis, i.e. for any three vectors in $\mathbb{R}^3$

$$A \times (B \times C) = B\,(A \cdot C) - C\,(A \cdot B)$$

(ii) identity − $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$

Since, to compute derivatives at a specific s (or t), one needs to know the (behavior of the) curve only near s (or t), it follows that the formulas in Exercise 3 are <u>local formulas</u>.