PSK 769R
F 2012

Lecture 1b

A defect of the Frenet–Serret frame is the need for nondegeneracy. There is an alternative way to frame a curve without requiring nondegeneracy.

At $s=0$ let $T(0)^{\perp}$ denote the plane normal to $T(0)$, and similarly $T(s)^{\perp}$ denotes the plane normal to $T(s)$. Pick a basis $\{M_1(0), M_2(0)\}$ for $T(0)^{\perp}$ such that $\{T(0), M_1(0), M_2(0)\}$ constitutes a right-handed orthonormal triad. Our goal is to propagate this triad to $\{T(s), M_1(s), M_2(s)\}$ in such a way that certain natural condition(s) hold.

right handedness $\Rightarrow$ $M_2(0) = T(0) \times M_1(0)$

$$M_2(s) = T(s) \times M_1(s)$$

Since $T(s) \cdot T(s) \equiv 1$, $T'(s) \in T(s)^{\perp}$. Hence there must exist $k_1(s)$, $k_2(s)$ such that

$$T'(s) = k_1(s) M_1(s) + k_2(s) M_2(s)$$

A unit vector field $M(s) \in T(s)^{\perp}$ is said to be <u>relatively parallel</u> along γ provided

$$M'(s) = f(s)\, T(s)$$

i.e. the vector M <u>turns as little as possible</u>. This is the natural condition we are looking for. We propagate $M_1(0), M_2(0)$, along the curve γ such that they remain relatively parallel at each s. Thus we require

$$M_1'(s) = f_1(s)\, T(s)$$

$$M_2'(s) = f_2(s)\, T(s)$$

for some as yet undetermined $f_i(\cdot)$, $i = 1, 2$.

But $\quad M_i(s) \cdot T(s) \equiv 0$

$$\Rightarrow \quad M_i'(s) \cdot T(s) + M_i(s) \cdot T'(s) \equiv 0$$

$$\Leftrightarrow \quad f_i(s) + k_i(s) \equiv 0$$

$$\Leftrightarrow \quad f_i(s) = -k_i(s) \qquad i = 1, 2.$$

<u>Definition</u> An orthonormal triad $\{T(s), M_1(s), M_2(s)\}$ is a <u>relatively parallel adapted frame</u> (RPAF) along a curve $s \mapsto r(s)$, if there exist <u>curvature functions</u> $k_1(\cdot)$, $k_2(\cdot)$, such that

$$T(s) = r'(s)$$

$$T'(s) = k_1(s)\, M_1(s) + k_2(s)\, M_2(s)$$

$$M_1'(s) = -k_1(s)\, T(s)$$

$$M_2'(s) = -k_2(s)\, T(s).$$

<u>Theorem</u> Given a C^2 curve $s \mapsto r(s)$, and a choice $M_1(0)$, $M_2(0)$ in $T(0)^\perp$ such that $\{T(0), M_1(0), M_2(0)\}$ is a right-handed orthonormal triad, there exists a unique RPAF along r that agrees with the initial choice.

<u>Proof</u>: $$M_1(s) = M_1(0) + \int_0^s M'(\sigma)\, d\sigma$$

$$= M_1(0) - \int_0^s k_1(\sigma)\, T(\sigma)\, d\sigma$$

$$k_1(s) = T'(s) \cdot M_1(s)$$

$$= T'(s) \cdot M_1(0) - \int_0^s k_1(\sigma)\, T'(s) \cdot T(\sigma)\, d\sigma$$

$$= \gamma''(s) \cdot M_1(0) - \int_0^s k_1(\sigma)\, \gamma''(s) \cdot \gamma'(\sigma)\, d\sigma$$

Similarly,

$$k_2(s) = \gamma''(s) \cdot M_2(0) - \int_0^s k_2(\sigma)\, \gamma''(s) \cdot \gamma'(\sigma)\, d\sigma$$

These are two uncoupled Voltera integral equations of the second kind. By the standard theory* of such integral equations, there exist unique $k_i(\cdot)$ $i=1,2$, solving the above equations ∎ There is a need to resort to numerical computation to solve the integral equations for a general curve γ, to determine natural curvatures k_1, k_2.

* M. Bocher (1909). An Introduction to the Study of Integral Equations; Cambridge Tracts in Mathematics and Mathematical Physics, No. 10.

We now have a generative model for the curve $s \longmapsto r(s)$ in terms of the RPAF associated to the initial frame and specified / programmed curvature functions $k_i(s)$, $i = 1, 2$,

$$\frac{d}{ds}\left[\begin{array}{ccc|c} T & M_1 & M_2 & r \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} T & M_1 & M_2 & r \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc|c} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[T(0)\ \ M_1(0)\ \ M_2(0)\right] \text{ prescribed.}$$

Relation of RPAF to Fre'net-Serret

Since $N(s)$, $B(s)$ are in $T^{\perp}(s)$ spanned by $M_1(s)$, $M_2(s)$,

$$\begin{aligned} N(s) &= \frac{1}{\kappa(s)} T'(s) \\ &= \frac{1}{\kappa(s)}\big(k_1(s) M_1(s) + k_2(s) M_2(s)\big). \end{aligned}$$

Then

$$1 \equiv N(s)\cdot N(s)$$

$$= \frac{k_1^2(s) + k_2^2(s)}{\kappa^2(s)}$$

$$\Rightarrow \quad \kappa(s) = \left(k_1^2(s) + k_2^2(s)\right)^{1/2}$$

$$B(s) = T(s) \times N(s)$$

$$= T(s) \times \left(\frac{k_1(s)}{\kappa(s)} M_1(s) + \frac{k_2(s)}{\kappa(s)} M_2(s)\right)$$

$$= -\frac{k_2(s)}{\kappa(s)} M_1(s) + \frac{k_1(s)}{\kappa(s)} M_2(s)$$

$$\tau(s) = -B'(s)\cdot N(s) \qquad \text{(torsion)}$$

$$= -\left(-\frac{k_2}{\kappa} M_1 + \frac{k_1}{\kappa} M_2\right)' \cdot \left(\frac{k_1}{\kappa} M_1 + \frac{k_2}{\kappa} M_2\right)$$

$$= \frac{1}{\kappa^2}\left(k_2' k_1 - k_1' k_2\right) \qquad \text{(work out missing step)}$$

$$= \left(\tan^{-1}\left(\frac{k_2}{k_1}\right)\right)'$$

$$= \theta'$$

where θ is polar angle in (k_1, k_2) plane, well-defined for $\varkappa > 0$.

From the integration

$$\theta(s) = \theta(0) + \int_0^s \tau(\sigma)\, d\sigma$$

it is clear that torsion gets accumulated in the polar angle. Since

$$N(s) = \cos(\theta(s))\, M_1(s) + \sin(\theta(s))\, M_2(s)$$

and

$$B(s) = -\sin(\theta(s))\, M_1(s) + \cos(\theta(s))\, M_2(s)$$

it is clear that $\theta(s)$ is the accumulated rotation (phase shift) of $\{N(s), B(s)\}$ relative to $\{M_1(s), M_2(s)\}$ as one proceeds along the curve from 0 to s.

The plane of (k_1, k_2) is called the plane of <u>normal development</u>. As $r(s)$ evolves in 3-D, the curve of normal development $(k_1(s), k_2(s))$ presents a picture that captures the essential 3-D ness of the curve $s \mapsto r(s)$.

Exercise 4.

Suppose the curve $s \mapsto r(s)$ is confined to a plane perpendicular to vector μ, not necessarily through the origin. Then the normal development is confined to a straight line passing through the origin in the (k_1, k_2) plane.

Exercise 5.

Suppose the curve $s \mapsto r(s)$ is confined to a sphere of radius R centered at $p \in \mathbb{R}^3$. Then the normal ~~devep~~ development is confined to a straight line in the (k_1, k_2) plane at a distance $\frac{1}{R}$ from $(0,0)$.

Remark One should think of the normal development of $s \mapsto r(s)$ as a signature curve