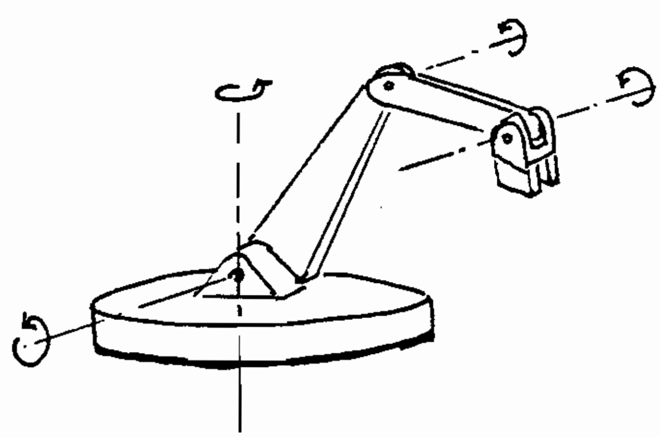
1. Kinematics of Manipulation

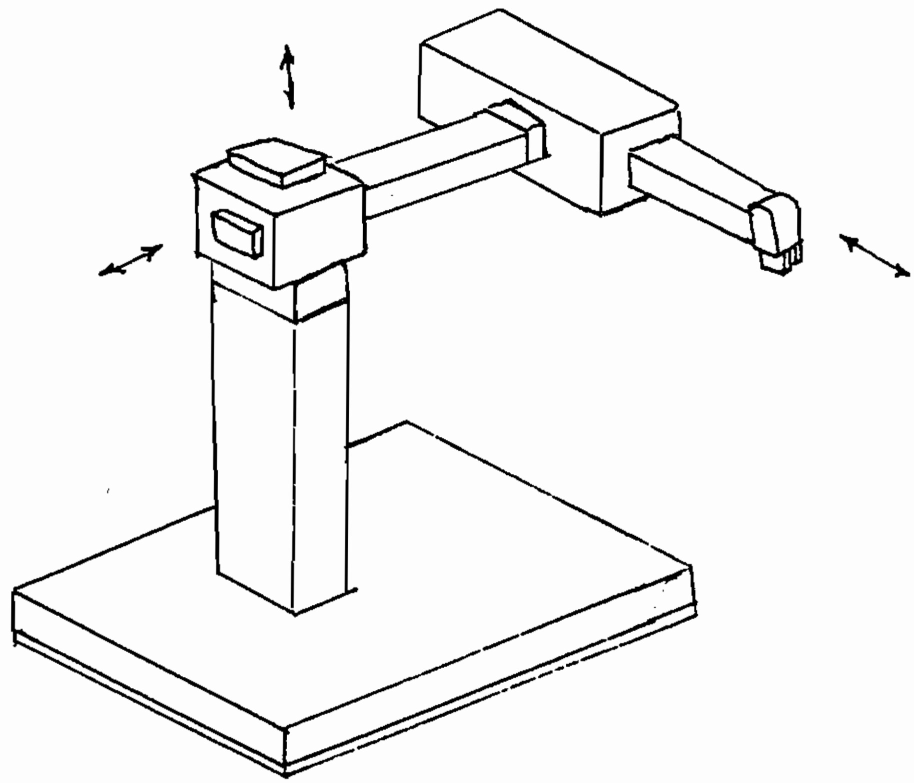
1.0 Introduction: Kinematics is the study of motion (Ampere coined the word from the Greek "$\kappa\iota'\nu\eta\mu\alpha$" meaning movement), and as such its application is not limited to mechanisms. In contrast, dynamics is the study of motion of a particle or a system of mass particles (rigid body, elastic plate, fluid in open channel etc.), induced by forces (and torques). In the context of robotic manipulation, we are interested in the kinematics and dynamics of linkages composed of interconnected rigid bodies. The joints are articulated and impose constraints on relative motions. We are interested in the **analysis** and **mathematical representation** of complex spatial motions executed by such a linkage subject to the **interconnection constraints**.

We will be mostly concerned with linkages in which the joints have either a single rotational degree of freedom (R pair) or a single translational degree of freedom (P pair). Robot manipulators will be composed of **links** which we will (for the most part) treat as rigid bodies. The links will often constitute **open** kinematic chains. The number of degrees of freedom of the robotic manipulator is thus equal to the number of joints. Closed chains are also of interest.

Example 1:



Example 2:



The standard dexterity or articulation requirement is six degrees of freedom, three degrees of freedom for positioning (translational) and three degrees of freedom for orientation (rotational). However, manipulators with many more degrees of freedom are found to be useful when operating in restricted work-spaces.

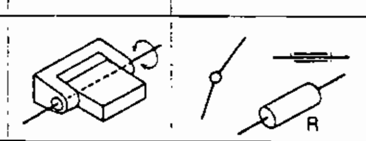
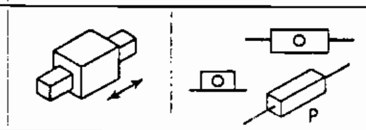
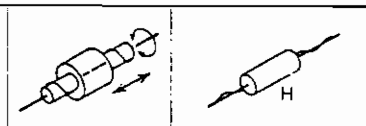
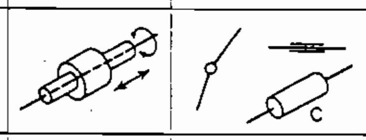
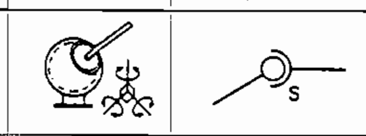
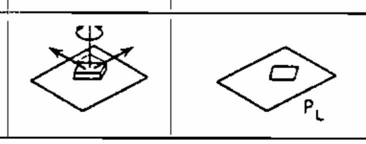
Examples 1 and 2 above show some typical arrangements of robot manipulator joints.

Classically, joints between rigid bodies are called **pairs** and are distinguished into two main classes:

(a) higher pairs - refer to contact occuring between a point and a surface, or a point and line or a line and a surface.

(c) lower pairs - contact does not occur in discrete points or lines. Instead it occurs between surfaces.

The lower kinematic pairs

Name of pair	Geometric form	Schematic representations	Relative degrees of freedom between elements of pair
Revolute (*R*)			1
Prismatic (*P*) (slide)			1
Helical (*H*) (screw)			1
Cylinder (*C*)			2
Sphere (*S*)			3
Plane (P_L) E			3

Among the various pairs sketched above, we distinguish the screw pair (H-pair) as having special significance. By definition, the **pitch** h is the distance the nut translates when the screw is rotated through 1 radian. When $h = \infty$ the H-pair becomes a P-pair. When $h = 0$ the H-pair becomes an R-pair. By convention h is taken as positive when the screw is right-handed and negative when it is left-handed.

If their contacting elements are suitably surfaced and properly lubricated, lower pairs form satisfactory working faces in machinery. The most common is the R-pair as a journal bearing. In machine tools we find the P-pair, as a slide-way and slider and also the H-pair. Marine thrust bearings use the E-pair; piston pumps use the C-pair and quite often the S-pair as well.

1.1 Rigid Motions

We introduce here the concept of rigid motion. Consider first a finite dimensional vector space V, over the real numbers. Let dim $(V) = n$. Let $\rho : V \times V \to \mathbb{R}_+$ be a metric on V. We say that a map ϕ: V$\to V$ **preserves** the metric ρ (or **leaves invariant** the metric ρ), if,

(1.1) $\rho(\phi(v), \phi(w)) = \rho(v, w)$ for all $v, w, \in V$.

Suppose, the vector space V carries an inner product $< \cdot, \cdot >$. Thus $< \cdot, \cdot > : V \times V \to \mathbb{R}$ is a map satisfying:

(a) $< v, w > = < w, v >$

(b) $< v_1 + v_2, w > = < v_1, w > + < v_2, w >$

(c) $< \alpha v, w > = \alpha < v, w >, \alpha \in \mathbb{R}$

(d) $< v, v > = 0 \Rightarrow v = 0$. Otherwise $< v, v > > 0$

Then we can associate a metric ρ by defining.

(1.2) $\rho(v, w) = < v - w, v - w >^{1/2}$

If $A : V \to V$ is an *invertible linear map* with the property that it leaves invariant the inner product, i.e. $< Av, Aw > = < v, w > \ \forall v, \ w \in V$, then A also leaves invariant the metric (1.2). We denote the collection of all such invertible linear maps as $O(V, < \cdot, \cdot >)$ or $O(V)$. If $A \in O(V)$ so does A^{-1}. Further the identity map $\mathbb{1} \in O(V)$. Also if $A_1, A_2 \in$ O(V), $A_1 \cdot A_2 \in O(V)$ where $A_1 \cdot A_2$ denotes the composition of linear maps. In other words, the set $O(V)$ has the structure of a group, the *orthogonal group* of the inner product space $(V, < \cdot, \cdot >)$. Now consider the collection of all *affine* transformations of V of the form $v \to Av + b$ where $b \in V$ and $A \in O\ (V)$. It follows that each such affine map also preserves the metric (1.2). This collection of affine maps is denoted as $E(V)$ and is called the group of rigid motions of $(V, < \cdot, \cdot >)$.

Clearly $O\ (V) \hookrightarrow E\ (V)$. The set $E\ (V)$ is also a group with the following properties;

if $\qquad g_1 v = A_1 v + b_1, \qquad g_1 \in E(V)$

$\qquad g_2 v = A_2 v + b_2, \qquad g_2 \in E(V)$

then $\qquad g_1^{-1} v = A_1^{-1} v - A_1^{-1} b_1 \qquad$ and,

$$(g_1 \, o g_2)v = (A_1 \, A_2)v \; + \; (A_1 \, b_2 \; + \; b_1).$$

We now consider the concrete situation when $V = \mathbb{R}^n$ and $< v, w >= \sum_{i=1}^{n} v_i \, w_i$, where v, w, $\in$ $\mathbb{R}^n$.

Then we denote the corresponding orthogonal group as $O(n)$. It is easy to see that,

$$O(n) \; = \; \left\{ \begin{matrix} A: \; A & n \times n \text{ matrix} \\ & A'A \; = \; I \end{matrix} \right\}. \qquad \text{(Here} \quad {}'\text{denotes transpose)}$$

Now, if we let $A \in O(n)$, then

$$1 = \; \det \, (A'A) = \; \det \, (A') \; det \; \; (A) = [\det(A)]^2.$$

Hence det $(A) = \pm 1$. Define $SO(n) = \{A \in O(n) : \; \det \; (A) \; = \; 1\}$. We call $SO(n)$ the ***special orthogonal group*** of $n \times n$ matrices. It is a subgroup of $O(n)$.

The rigid motion group of $\mathbb{R}^n$ with the standard Euclidean metric will be denoted as $E(n)$. It is possible to identify $\mathbb{R}^n$ with a subset of $\mathbb{R}^{n+1}$ by means of the imbedding

$$i : \; \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$$
$$x \mapsto \begin{pmatrix} x \\ ----- \\ 1 \end{pmatrix}.$$

We can then view the action of $E(n)$ on $\mathbb{R}^n$ as equivalent to the action on $i(\mathbb{R}^n)$ of the group of linear transformations of the form

$$\begin{bmatrix} A & | & b \\ - & | & - \\ 0 & | & 1 \end{bmatrix}$$

where $A \in O(n)$ and $b \in \mathbb{R}^n$. More precisely $E(n)$ has a representation as the group of matrices of the form

$$\begin{bmatrix} A & | & b \\ - & | & - \\ 0 & | & 1 \end{bmatrix} \; , \quad A \in O(n), \quad b \in \mathbb{R}^n \, .$$

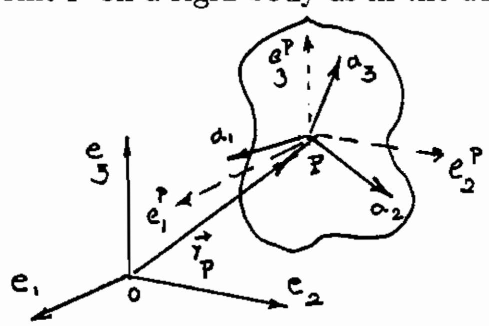
We make some dimension calculations. The equation $A'A \; = \; \mathbb{1}$ on $(n \times n)$ matrices corresponds to $n + \frac{n^2-n}{2} \; = \; \frac{(n+1)}{2}$ scalar equations. There are n^2 unknowns in A. Thus one expects $O(n)$ to be an $\frac{n(n-1)}{2}$ - parameter family of matrices. More precisely $O(n)$ is a differentiable manifold with 2 connected components each of dimension $\frac{n(n-1)}{2}$. Thus $O(3)$ is of dimension 3. We use the notation $SE(n)$ to denote the *Euclidean group*

$$\left\{ \begin{bmatrix} A & | & b \\ - & | & - \\ 0 & | & 1 \end{bmatrix} : \quad A \in SO(n), \quad b \in \mathbb{R}^n \right\}.$$

1.2 Rigid Motions in 3 Dimensions and Rigid Bodies

We now specialize the motions of the previous section to 3-dimensional Euclidean space. It should be emphasized that the groups O(3) and E(3) have additional roles as configuration spaces. This we explain below.

A rigid body is a system of particles with the property that the relative distances between the particles remain fixed under all motions / configurations of the body. Choose any point P on a rigid body as in the adjoining figure. With this point P as origin,



attach a *body fixed* orthonormal *frame* $\{a_1, a_2, a_3\}$. Let $\{e_1, e_2, e_3\}$ be an arbitrary orthonormal fixed frame. Then a *configuration* of the rigid body is completely specified by the following:

- The vector $\vec{r}_P = \overrightarrow{0P}$

- The 3x3 matrix $A_a \in SO(3)$ that carries the translated frame $\{e_1^P, e_2^P, e_3^P\}$ (where each e_i^P is parallel to e_i) to the body - fixed frame $\{a_1, a_2, a_3\}$.

More precisely,

$$\begin{aligned} A_a e_1^P &= a_1 \\ A_a e_2^P &= a_2 \\ A_a e_3^P &= a_3 \end{aligned} \tag{1.2.1}$$

Thus the matrix

$$\begin{bmatrix} A_a & | & r_P \\ - & | & - \\ 0 & | & 1 \end{bmatrix}$$

specifies a configuration of the rigid body. In other words the configuration space (= set of all possible configurations) of a rigid body can be identified with $SE(3)$. If further the rigid body is constrained to move in such a way that the body has a fixed point - which we may assume to be P, then by identifying that point P with the reference frame origin 0, we see that the configuration of the rigid body is completely determined by a matrix A_a (also known as the attitude matrix of the rigid body). We thus summarize that

(i) for a general rigid body the configuration space is $SE(3)$;

(ii) for a rigid body with a fixed point P, the configuration space is $SO(3)$.

From equation (1.2.1) it is obvious that,

$$A_a = [a_{ij}]$$

where,

(1.2.2)

$$\begin{aligned} a_{ij} &= < a_i, e_j^P > \\ &= \text{cosine of the angle between the unit vectors } a_i \text{ and } e_j^P \end{aligned}$$

Hence the matrix A_a is also called the *direction cosine matrix.*

We have already remarked that $O(3)$ and $SO(3)$ are three dimensional manifolds. One way to see this is through the so called Euler angle parametrization of the direction cosine matrix.

The Euler Angles of A_a:

The angular orientation of the frame $a \underline{\Delta}\{a_1, a_2, a_3\}$ fixed to the body relative to the frame $e^P \underline{\Delta}\{e_1^P,\ e_2^P,\ e_3^P\}$ is thought to be the result of three successive rotations. Before the first rotation, the frame $a\underline{\Delta}\{a_1,\ a_2,\ a_3\}$ coincides with the frame $e^P\underline{\Delta}\{e_1^P,\ e_2^P,\ e_3^P\}$. The first rotation is carried out about the axis e_3^P through the angle ψ. This carries the body fixed frame from its original orientation to an orientation denoted as $e^{P''}$. The second rotation is through an angle θ about the axis $e_1^{P''}$, results in the orientation denoted as $e^{P'}$.

The third rotation through an angle ϕ about the axis $e_3^{P'}$ produces the final orientation of the body. The angles ψ, θ, ϕ are known as the Euler angles. Let.

$$A^{\psi} = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.3}$$

$$A^{\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix} \tag{1.2.4}$$

and

$$A^{\phi} = \begin{bmatrix} cos(\phi) & sin(\phi) & 0 \\ -sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.5}$$

Then,

$$A_a = A^{\phi}A^{\theta}A^{\psi}. \tag{1.2.6}$$

If we use the notation,

$$\begin{aligned} cos(\theta) &= c_{\theta}, & cos\phi &= c_{\phi}, \\ cos(\psi) &= c_{\psi}, & sin(\theta) &= s_{\theta}, \\ sin(\phi) &= s_{\phi} \quad \text{and} & sin(\psi) &= s_{\psi}, \end{aligned}$$

then

$$A_a = \begin{bmatrix} c_{\psi}c_{\phi} - s_{\psi}c_{\theta}s_{\phi} & s_{\psi}c_{\phi} + c_{\psi}c_{\theta}s_{\phi} & s_{\theta}s_{\phi} \\ -c_{\psi}s_{\phi} - s_{\psi}c_{\theta}c_{\phi} & -s_{\psi}s_{\phi} + c_{\psi}c_{\theta}c_{\phi} & s_{\theta}c_{\phi} \\ s_{\psi}s_{\theta} & -c_{\psi}s_{\theta} & c_{\theta} \end{bmatrix} \tag{1.2.7}$$

The Euler angle parametrization is physically realized by the Cardan suspension in Fig. 1.2.1. The angles ψ, θ and ϕ are, in this order, the rotation angle of the outer gimbal relative to the material base, of the inner gimbal relative to the outer gimbal and of the body relative to the inner gimbal. For $\theta = n\pi (n = 0, 1, 2, ...)$, the two gimbals coincide (gimbal lock).

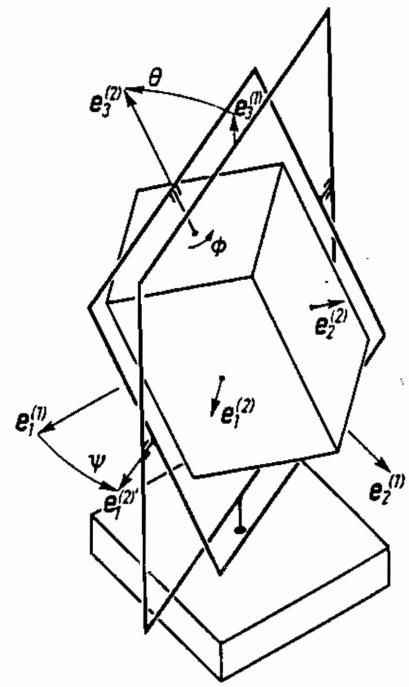


Fig. 1.2.1 Euler Angles in Cardan Suspension

A basic result on rigid motions is Euler's theorem.

Theorem (Euler):

Any motion of a rigid body with one point P fixed can be obtained by a pure (clockwise) rotation of the rigid body about an axis passing through the fixed point.

Proof: We have to show that given an element $A \in SO(3)$ there is a unit vector $c \in \mathbb{R}^3$ such that c is the axis of rotation (and hence fixed) and A is representable as a pure rotation about c.

$$\text{To see this, let } spec\ (A) \triangleq \text{set of eigenvalues of } A \ = \ \{\lambda_1, \lambda_2, \lambda_3\}$$

Since

$$\begin{aligned} Ax \ = \ \lambda x &\Longrightarrow A'Ax = \lambda A'x \\ &\Longrightarrow A'x = \frac{1}{\lambda}x \end{aligned}$$

it follows that $\lambda \ \in \ spec(A)$ iff $\lambda^{-1} \ \in \ spec\ (A') = spec(A)$.

In other words, we see that the eigenvalues of A occur in reciprocal pairs. Now, $\lambda_1\lambda_2\lambda_3 = det(A) \ = \ +1$. Suppose $\lambda_1 \neq 1$. Then since $\lambda_1^{-1} \in spec\ (A)$, one of the two eigenvalues λ_2, λ_3 must have the value λ_1^{-1}. Let us say $\lambda_2 = \lambda_1^{-1}$.

Then $1 = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_1^{-1} \lambda_3 = \lambda_3$.

Thus we see that $1 \in spec\,(A)$. Let c be a corresponding unit eigenvector; $Ac = c$. The vector c plays the role of the axis of rotation. Let d and e be two unit vectors spanning the 2-plane $c^\perp$. Then,

$$< Ad, c > = < Ad, Ac > = < d, c > = 0$$

$$< Ae, c > = < Ae, Ac > = < e, c > = 0.$$

Thus A leaves invariant the 2-plane $c^\perp$. Since A is length preserving it is a rotation in the plane $c^\perp$ ∎

Remark We can make a bit more explicit our arguments above. Choosing $\{c, d, e\}$ as basis, we are looking for the matrix representation of A in that basis. Suppose.

$$v = v_1 c + v_2 d + v_3 e \qquad v_i \in \mathbb{R}.$$

Then

$$\begin{aligned} Av &= A(v_1 c + v_2 d + v_3 e) \\ &= v_1 Ac + v_2 Ad + v_3 Ae \\ &= v_1 \cdot (a_{11} c + a_{21} d + a_{31} e) + v_2 \cdot (a_{12} c + a_{22} d + a_{32} e) + v_3 \cdot (a_{13} c + a_{23} d + a_{33} e). \\ &= (\sum_{j=1}^{3} a_{1j} v_j) c + (\sum_{j=1}^{3} a_{2j} v_j) d + (\sum_{j=1}^{3} a_{3j} v_j) e. \end{aligned}$$

The matrix $[a_{ij}]$ is the matrix representation of $A \in SO(3)$ in the new basis.

Since $Ac = c$, we have

$$a_{11} = 1\ ,\ a_{21} = 0,\ a_{31} = 0;$$

also

$$\begin{aligned} 0 &= < c,\ d > = < Ac,\ Ad > = < c,\ Ad > = a_{12}; \\ 0 &= < c,\ e > = < Ac,\ Ae > = < c, Ae > = a_{13}; \\ 1 &= < d,\ d > = < Ad,\ Ad > = (a_{22})^2 + (a_{32})^2 \\ 1 &= < e,\ e > = (a_{23})^2 + (a_{33})^2 \\ 0 &= < d,\ e > = < Ad,\ Ae > = a_{22}\, a_{23} + a_{32}\, a_{33} \end{aligned}$$

Thus the matrix representation of A **in the basis** $\{c, d, e\}$ is of the form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \tag{1.2.8}$$

where

$$\begin{aligned} (a_{22})^2 + (a_{23})^2 &= 1 \\ (a_{32})^2 + (a_{33})^2 &= 1 \\ a_{22}a_{32} + a_{23}a_{33} &= 0. \end{aligned} \tag{1.2.9}$$

It can be immediately verified that $a_{22} = cos(\phi)$, $a_{23} = sin(\phi)$, $a_{32} = -sin(\phi)$, $a_{33} = cos(\phi)$ is a solution of (1.2.9) for suitable ϕ. ϕ is essentially unique. In fact if we observe that tr(A) is independent of the choice of basis then we see that,

$$2cos(\phi) + 1 = tr(A)$$

or

$$\phi = cos^{-1}[\frac{1}{2}\{tr(A) - 1\}].$$

This leads us to,

EULER'S theorem (alternative form)

Let $A \in SO(3)$. Then, there exists c such that $c'c = 1$, $Ac = c$ and,

$$\begin{aligned} A &= exp[\phi S(c)] \\ &= I + sin(\phi) \cdot [S(c)] \ + (1 - cos(\phi)) \cdot S^2(c) \end{aligned}$$

where, by $S(c)$ we mean the skew-symmetric matrix

$$S(c) = \begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{bmatrix}$$

and we also use the identity,

$$S^2(c) = -I + cc'$$

The proof of this alternative form of Euler's theorem is left as an *exercise*.

Another way to think about the parameterization problem for $SO(3)$ is to use the ***Cayley transform***. Let $A \in SO(3)$. Then, for any $x \in \mathbb{R}^3$, $< Ax,\ Ax > = < x,\ x >$.

Let $y = Ax$. Then,

$$
\begin{aligned}
0 &= < y,\ y > - < x,\ x > \\
&= < y - x,\ y + x > \\
&= < (A - \mathbb{1})x,\ (A + \mathbb{1})x > .
\end{aligned}
$$

Suppose -1 is ***not*** an eigenvalue of A. Then $(A + \mathbb{1})$ is invertible. Let.

$$f = (A + \mathbb{1})x.$$

As x varies over all of $\mathbb{R}^3$, so does f. Further,

$$< (A - \mathbb{1})\,(A + \mathbb{1})^{-1}f,\ f > = 0 \quad \forall f \in \mathbb{R}^3 .$$

It then follows that,

$$B = (A - \mathbb{1})\,(A + \mathbb{1})^{-1}$$

is a skew-symmetric matrix. Further, letting,

$$B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

we conclude that,

$det\,(\mathbb{1} - B) = 1 + b_1^2 + b_2^2 + b_3^2 > 0$. Hence $(\mathbb{1} - B)$ is invertible. Therefore,

$$A = (\mathbb{1} - B)^{-1}\,(\mathbb{1} + B).$$

This is known as the ***Cayley transform***. We have shown that any 3×3 orthogonal matrix A can be written as a Cayley transform of a skew- symmetric matrix provided $-1 \notin spec\,(A)$. Note further that,

$det\left((\mathbb{1} - B)^{-1}\,(\mathbb{1} + B)\right)$

$$
\begin{aligned}
&= det\left((\mathbb{1} - B)^{-1}\right)\ det\ (\mathbb{1} + B)\\
&= \left(det\ (\mathbb{1} - B)\right)^{-1}\ det\ (\mathbb{1} + B)'\\
&= \left(det\ (\mathbb{1} - B)\right)^{-1}\ det\ (\mathbb{1} - B)\\
&= 1.
\end{aligned}
$$

Thus the range of the Cayley transform is $\subset\ SO(3)$.

Exercise: Show that $\varphi(T'\ B\ T) \ = \ T'\varphi(B)T$ for any skew symmetric B and orthogonal matrix T, where $\varphi(\cdot)$ denotes the Cayley transform.

Remark: The parameters $(b_1,\ b_2,\ b_3)$ are called Rodrigues parameters.

Our next result deals with a general representation of elements of $SE(3)$. This result, attributed to Michel Chasles (1793-1880), says that the most general rigid motion in $\mathbb{R}^3$ is a *helical (or screw)* motion. More precisely,

Theorem [Chasles]:

Given

$$
\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}
$$

from SE(3), there exists $p \in \mathbb{R}^3$ such that,

$$
(*) \qquad \begin{bmatrix} \mathbb{1} & -p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{1} & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} M & c \\ 0 & 1 \end{bmatrix}
$$

Where $\mathbb{1}$ denotes the 3×3 identity matrix, c is a vector such that $Ac = c$ and $M = \exp[\phi S(\tilde{c})]$ where $\tilde{c} \ = \ c/\|c\|$.

Proof: For simplicity we shall assume that,

$$
(H) \qquad Ker(A - \mathbb{1}) \ = \ \{x : \quad (A - \mathbb{1})x \ = \ 0\}
$$

is a one-dimensional vector space.

The left hand side of $(*)$ is equal to

$$
\begin{bmatrix} A & Ap + b - p \\ 0 & 1 \end{bmatrix}.
$$

Thus, from (*)

$$A = M \qquad \text{and} \qquad Ap + b - p = c$$

$$\text{or } (\mathbb{1} - A)p + c = b$$

The problem thus reduces to finding p, c such that

$$(1.2.10) \qquad \begin{aligned} (\mathbb{1} - A)p + c &= b \\ (\mathbb{1} - A)c &= 0 \end{aligned}$$

aside: the Fredholm alternative says that a linear equation.

$$Qv = h$$

has a solution *iff* for all vectors w satisfying $Q'w = 0$, we have $w'h = 0$.

Apply Fredholm alternative to (1.2.10).

Equation (1.2.10) has solution $\begin{pmatrix} p \\ c \end{pmatrix}$ iff for all $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying,

$$(1.2.11) \qquad \begin{bmatrix} (\mathbb{1} - A)' & 0 \\ \mathbb{1} & (\mathbb{1} - A)' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we have

$$\begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} b \\ 0 \end{pmatrix} = x'b = 0.$$

Now,

$$(1.2.12) \qquad (\mathbb{1} - A)'x = 0$$

$$(1.2.13) \qquad x + (\mathbb{1} - A)'y = 0$$

$\Rightarrow x = \alpha c$ for same real number $\alpha \in \mathbb{R}$.

Furthermore, equation (1.2.13) is equivalent to

$$(1.2.14) \qquad (\mathbb{1} - A)'y = -\alpha c.$$

Apply Fredholm alternative to (1.2.14). Thus $(\mathbb{1}-A)'y = -\alpha c$ has a solution *iff* for every z such that,

$$(\mathbb{1}-A)z = 0,$$

it follows that $< z, -\alpha c >= 0$.

But $(\mathbb{1}-A)z = 0 \Rightarrow z = \beta c$ for some $\beta \in \mathbb{R}$. Hence, $< \beta c, -\alpha c >= 0 \quad \forall \beta \in \mathbb{R}$. This is possible only if $\alpha = 0$. Hence,

$$Ker\left\{\begin{bmatrix} (\mathbb{1}-A)' & 0 \\ \mathbb{1} & (\mathbb{1}-A)' \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 \\ \beta c \end{bmatrix} : \beta \in \mathbb{R}\right\} \tag{1.2.15}$$

Clearly

$$Ker\begin{bmatrix} (\mathbb{1}-A)' & 0 \\ \mathbb{1} & (\mathbb{1}-A)' \end{bmatrix} \perp \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Thus we see that (1.2.10) is solvable and the proof of the theorem is complete.■

However, for our purposes it is necessary to find the interpretation of this result in terms of screw motions.

Since $Ker(\mathbb{1}-A)$ is assumed to be one dimensional, any c satisfying $(\mathbb{1}-A)c = O$ can be written as $c = \alpha v$ where v is a unit vector. Thus, suppose

$$\begin{pmatrix} p_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} p_1 \\ \alpha_1 v \end{pmatrix}$$

and

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} p_2 \\ \alpha_2 v \end{pmatrix}$$

are two solutions to (1.2.10). We see that

$$(\mathbb{1}-A)(p_1 - p_2) = (\alpha_2 - \alpha_1)v$$

where $v \in Ker(\mathbb{1}-A)$. We can verify that this equation has a solution *iff* $\alpha_2 - \alpha_1 = 0$ in which case

$$c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = c_2 = \gamma v,$$

and

$$p_1 - p_2 = \alpha v$$

or

$$p_1 = p_2 + \alpha v.$$

Thus the end points of all the vectors p_1, p_2, p_3... lie on a certain distinguished line whose direction is determined by the vector c. This is called the *screw axis*.

How do points on the screw axis transform?

Let x be a point on the screw axis. Then $x \mapsto Ax + b$.

Now $x = p + \alpha c$ for some $\alpha \in \mathbb{R}$ and c, p satisfying (1.2.10)

$$\begin{aligned} Ax + b &= A(p + \alpha c) + b \\ &= Ap + \alpha Ac + b \\ &= Ap + \alpha c + b \\ &= p + c + \alpha c \\ (\textit{from } 1.2.10) \qquad &= p + (\alpha + 1)c \end{aligned}$$

Thus *any* point on the screw axis gets merely *translated* along the axis a distance

$$\begin{aligned} &= \|p + (\alpha + 1)c - (p + \alpha c)\| \\ &= \|c\| \end{aligned}$$

What about points away from the screw axis?

Suppose,

$$x = p + \alpha c + w \neq 0$$

Then

$$\begin{aligned} Ax + b &= A(p + \alpha c + w) + b \\ &= (Ap + \alpha c + b) + Aw \\ &= p + (\alpha + 1)c + Aw \end{aligned}$$

What if $x = 0$?

$$\begin{aligned} A0 + b = b = (\mathbb{1} - A)p + c &= p + \alpha c + c + Aw \\ &= (p + \alpha c) + (Aw + c) \end{aligned}$$

Points away from the screw axis thus undergo both rotation and translation in general.

What is the pitch? (Symbol h is used for pitch).

By definition, the pitch h is the ratio = translational motion along screw axis / rotation in radians. In the present case,

$$h = pitch = \frac{\|c\|}{\phi} = \frac{\|c\|}{\cos^{-1}\left\{\frac{1}{2}\left[tr\ (A)\ -\ 1\right]\right\}}$$

If $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ is a pure translation, then $A = \mathbb{1}$. We then say that pitch $= \infty$. If $b = 0$ *then (1.2.10) has the only solution* $p = 0 = c$. Thus pitch $= \|c\|/\phi = 0/\varphi = 0$.

We choose to write the r.h.s of the statement of Chasles theorem in the form.

(1.2.16)
$$\begin{bmatrix} exp[\phi S(v)] & hv \\ 0 & 1 \end{bmatrix}$$

Where v is the *unit* vector satisfying $Av = v$ and $h =$ pitch. The matrix in (1.2.16) is known as the *screw matrix* associated to the given element of SE(3).

The screw matrix is completely specified by specifying ϕ, h and the 2 parameters determining the unit vector v. To specify $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ completely we need in addition to specify the vector p.

However since p is constrained to lie on a line (the screw axis), we only obtain 2 more parameters. Thus, Chasles' theorem gives us a particular choice of *six parameters* for determining an element of $SE(3)$.

We will see in the next section that a line in 3 dimensional space is completely determined by 4 parameters called the true line coordinates, so we can also view Chasles' theorem as saying that $SE(3)$ is parametrized by;

4	true coordinates of the screw axis
1	pitch parameter h
1	angle parameter ϕ

1.3 Homogenous Coordinates:

A Points: The position of a point in 3 dimensional space can be uniquely defined by the ratios of *four* coordinates x, y, z, t and conversely, these ratios define a point uniquely. We assume that not all x, y, z, t are simultaneously zero. One can identify the cartesian coordinates as $(\frac{x}{t}, \frac{y}{t}, \frac{z}{t})$ assuming $t \neq 0$. The point defined by $(x, y, z, 0)$ is considered to be the point at infinity and is denoted as ∞. Clearly (x, y, z, t) and $(\lambda x, \lambda y, \lambda z, \lambda t)$ denote the same point for all $\lambda \neq 0$. This is the extent of nonuniqueness in the homogeneous coordinates (x, y, z, t). We often use the notation $P(x, y, z, t)$ to donate a point P with homogeneous coordinates (x, y, z, t).

Suppose A (x_1, y_1, x_1, t) and B (x_2, y_2, z_2, t) are two given points. We define the line AB to consist of the points P(x, y, z, t) for which values of the ratio λ/μ can be found such that

(1.3.1)
$$x = \lambda x_1 + \mu x_2; \qquad y = \lambda y_1 + \mu y_2;$$
$$z = \lambda + \mu z_2; \qquad t = \lambda t_1 + \mu t_2.$$

Every value of λ/μ determines one and only one point, which is called a point of the line AB. In particular $\mu = 0$ determines A and $\lambda = 0$ determines B. The two values λ, μ cannot vanish simultaneously. In a compact form we can write the previous system of 4 equations as;

(1.3.2)
$$\mathbf{P} = \lambda \mathbf{A} + \mu \mathbf{B}$$

or more symmetrically,

(1.3.3)
$$\lambda \mathbf{A} + \mu \mathbf{B} + \nu \mathbf{P} = 0$$

(note: an expression of the form $\lambda_1 \mathbf{P}_1 + ... + \lambda_n \mathbf{P}_n = 0$ is called a *syzygy*).

Let $A(x_1, y_1, z_1, t_1)$, $B(x_2, y_2, z_2, t_2)$ and $C(x_3, y_3, z_3, t_3)$ be three given non-colinear points (thus they are *not in syzygy).* We define the plane ABC to consist of the points $P(x, y, z, t)$ for which values of the ratios $\lambda : \mu : \nu$ can be found such that:

$$x = \lambda x_1 + \mu x_2 + \nu x_3$$

$$y = \lambda y_1 + \mu y_2 + \nu y_3$$

(1.3.4)
$$z = \lambda z_1 + \mu z_2 + \nu z_3$$

$$t = \lambda t_1 + \mu t_2 + \nu t_3.$$

Every set of values $\lambda : \mu : \nu$ (not all zero) determines one and only one point which is called a point of the plane ABC. In particular, $\mu = \nu = 0$ determines A. $\lambda = \nu = 0$ determines B, and $\mu = \lambda = 0$ determines C. The corresponding syzygy is:

(1.3.5)
$$\mathbf{P} = \lambda \mathbf{A} + \mu \mathbf{B} + \nu \mathbf{C}.$$

The coordinates of P satisfy the determinantal relation:

(1.3.6)
$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ t & t_1 & t_2 & t_3 \end{vmatrix} = 0$$

B Line Geometry: Let P_1 (x_1, y_1, z_1, t_1) and P_2, (x_2, y_2, z_2, t_2) be two distinct points. We define from their coordinates the six numbers,

(1.3.7)
$$l = x_1 t_2 - x_2 t_1; \; m = y_1 t_2 - y_2 t_1; \; n = z_1 t_2 - z_2 t_1$$

$$l' = y_1 z_2 - y_2 z_1; \; m' = z_1 x_2 - z_2 x_1; \; n' = x_1 y_2 - x_2 y_1.$$

We prove first that the ratios of these six numbers are unaltered if the points $\mathbf{P_1}$ and $\mathbf{P_2}$ are replaced by *any two* points $\tilde{P}_1$ and $\tilde{P}_2$ of the line $\mathbf{p}$ joining them.

In fact, if:

$$\tilde{\mathbf{P}}_1 = \lambda\tilde{\mathbf{P}}_1 + \mu\tilde{\mathbf{P}}_2 \tag{1.3.8}$$

$$\tilde{\mathbf{P}}_2 = \nu\tilde{\mathbf{P}}_1 + \delta\tilde{\mathbf{P}}_2$$

then it is easy to show that,

$$(\tilde{x}_1\tilde{t}_2 - \tilde{x}_2\tilde{t}_1) = (\lambda\delta - \mu\nu)(x_1t_2 - x_2t_1) \tag{1.3.9}$$

$$(\tilde{y}_1\tilde{z}_2 - \tilde{y}_2\tilde{z}_1) = (\lambda\delta - \mu\nu)(y_1z_2 - y_2z_1)$$

etc., and the result is immediate.

Thus the ratios of the six numbers l, m, n, l', m', n' are suitable as coordinates for a line, because:

(i) if the line is given, the ratios of the numbers are determined.

(ii) if the ratios are given then the line is determined uniquely.

We therefore call these numbers the coordinates of the line. They are also called the ***Plücker*** coordinates of the line after the German geometer *Julius* ***Plücker*** who invented them in 1865.

Note that the six coordinates of a line cannot be taken at random. They must satisfy the relation.

$$ll' + mm' + nn' = 0. \tag{1.3.10}$$

Proof of Relation 1.3.10:

Suppose that one line p is defined by two points P_1 (x_1, y_1, z_1, t_2) and P_2 $(x_2\ y_2,\ z_2,\ t_2)$ another line q is defined by the points Q_1 $(X_1, Y_1, Z_1, T_1,)$ and Q_2 (X_2, Y_2, Z_2, T_2). The Plucker coordinates for p and q are respectively $(\ell, m, n, \ell', m', n')$ and (L, M, N, L', M', N'). It can be verified that

$$\begin{vmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ X_1 & Y_1 & Z_1 & T_1 \\ X_2 & Y_2 & Z_2 & T_2 \end{vmatrix} = lL' + l'L + mM' + m'M + nN' + n'N \tag{1.3.11}$$

Now set $P_1 = Q_1$ and $P_2 = Q_2$. Then the left hand side of (1.3.11) in zero since two pairs of rows the matrix are equal. The r.h.s. of (1.3.11) is $2(ll' + mm' + nn')$. Hence we have shown that 1.3.10 holds.■

Theorem 1.3.1: If two lines whose coordinates are (l, m, n, l', m', n') and (L, M, N, L', M', N') intersect, then $lL' + l'L + mM' + m'M + nN' + n'N = 0$.

Proof: If the lines intersect the points P_1, P_2, Q_1, Q_2 defined above are coplanar and the result follows from equation (1.3.6). The converse of theorem 1.3.1 is also true. The proof is left to the reader.■

Definition 1.3.1: Given two lines $(l_1, m_1, n_1, l_1', m_1', n_1',)$ and $(l_2, m_2, n_2, l_2', m_2', n_2')$ we call the quantity $w_{12} = l_1 l_2' + l_2 l_1' + m_1 m_2' + m_2 m_1' + n_1 n_2' + n_2 n_1')$, the **mutual invariant** of the two lines.

This quantity w_{12} has a significant physical interpretation (*as virtual work)* in our later discussion of equilibrium analysis of rigid bodies subject to external forces and torques.

Remark (1.3.1) The coordinates (l, m, n, l', m', n') and $(\lambda l, \lambda m, \lambda n, \lambda l', \lambda m', \lambda n')$ denote the same line. This together with (1.3.10) implies that there is a 4 parameter family of lines in 3-space.

Remark (1.3.2) For a line (l, m, n, l', m', n') to be finite, at least one of l, m, n must be nonzero. Otherwise the line is said to be at ∞ . When l', m', n' all vanish, the line passes through the origin. The line at ∞ perpendicular to the z axis is given by $(0, 0, 0, 0, 0, 1)$. Given a finite line (a, b, c, d, e, f) the line at ∞ perpendicular to this line is given by $(0, 0, 0, a, b, c,)$.

Remark (1.3.3) From eqn. (1.3.10) it follows that a line in 3-space can be viewed as a point of a quadric in 5-dimensional space. This is the famous *Plücker* quadric.

C. A line as the intersection of two planes

A line may be defined as the intersection of 2 planes.

$$P_1 : a_1x + b_1y + c_1z + d_1t = 0$$

$$P_2 : a_2x + b_2y + c_2z + d_2t = 0$$

Then (the) line coordinates of the line of intersection of the two planes P_1 and P_2 is given by the relations:

$$l = b_1c_2 - b_2c_1;\ m = c_1a_2 - c_2a_1;\ n = a_1b_2 - a_2b_1;$$

$$l' = a_1d_2 - a_2d_1;\ m' = b_1d_2 - b_2d_1;\ n' = c_1d_2 - c_2d_1$$

D. Interpretation of Plücker Coordinates

Consider the line in 3-space as in the adjoining figure. The line segment $\vec{\rho} = PQ$ can be said to have a *moment* about the origin 0 as $\vec{\rho} \times \vec{V} = \vec{\rho} \times \vec{l}$ where $\vec{l} =$ directed $\perp$ from 0 to the line. If $\underline{P} = (x_1, y_1, z_1, t_1)$ and $Q = (x_2, y_2, z_2, t_2)$ then the formula (1.3.7) determines (up to a scale factor) the vector $\vec{\rho} \sim (l, m, n)$ and its moment $\vec{\rho} \times \vec{l} \sim (l', m', n')$. This can be easily checked from the definition of the vector cross-product and is left to the reader.