1.4 Kinematics of Linkages

A first step in understanding the kinematics of linkages is to determine a systematic way of assigning coordinate systems to the links. The convention by which this is usually carried out is due to J. Denavit and R.S. Hartenberg and appears in their 1955 paper - "A kinematic notation for lower pair mechanisms based on matrices", Trans ASME 77E (J. App. Mech. 22) pp. 215-221. We explain this below

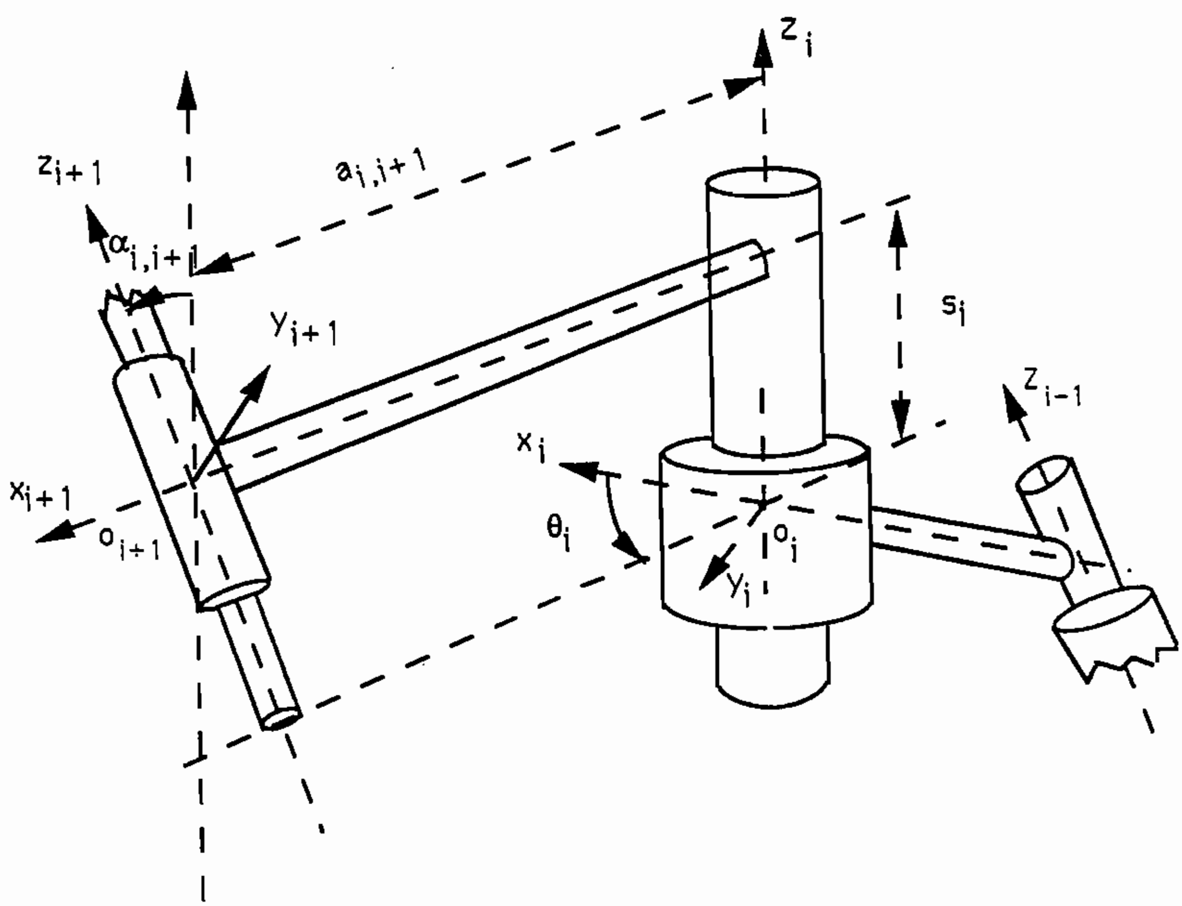


Fig: 1.4.1

Generalized lower pair with axis $O_i z_i$; its neighboring pairs. and the quantities θ_i, s_i, $\alpha_{i,i+1}, a_{i,i+1}$ required when transforming coordinates.

Consider the i^{th} pair in a linkage-chain, and align its axis along the z_i axis. The $x_i y_i z_i$ system is determined by directing the x_i- axis along the common perpendicular from the

$(i-1)$th pair. The process is repeated from the i^{th} pair to define the $(i+1)^{th}$ pair and its $(x_{i+1}, y_{i+1}, z_{i+1})$ coordinate system. To transfer from the i-system to the $(i+1)$-system four quantities in carefully consistent senses and directions must be specified;

(a) a twist $\alpha_{i,i+1}$ about the x_{i+1} axis and a distance $a_{i,i+1}$ parallel to it;

(b) a rotation θ_i about the z_i axis and a translation s_i parallel to it.

The four quantities $\theta_i, s_i, \alpha_{i,i+1}$, and $a_{i,i+1}$ uniquely determine the relative positions of two adjacent pair axes.

While $\alpha_{i,i+1}$ and $a_{i,i+1}$ are always fixed (and are commonly called the 'twist' and 'length' respectively of the link $(i, i+1)$), θ_i and s_i can, one or both of them, vary according to the nature of the pair whose axis lies along z_i. If this pair is a C-pair then θ_i and s_i are variables that are independent of each other; if it is an R-pair then θ_i varies and s_i is a fixed **offset** dimension; if it is a P-pair θ_i is fixed and s_i is a variable translation: if it is an H-pair a further condition $\delta s_i = h\delta\theta_i$ needs to be specified where h is the pitch of the screw defining the H-pair. An S-pair can be represented as three **cointersecting** series-connected R-pairs, each one preferably at right angles to its neighbor. A 4×4 matrix representing the transfer of coordinates from the $(i+1)$-system to the i-system is given by,

$$A_{i,i+1} = \begin{bmatrix} \cos(\theta_i) & -sin(\theta_i)cos(\alpha_{i,i+1}) & sin(\theta_i)sin(\alpha_{i,i+1}) & a_{i,i+1}cos(\theta_i) \\ sin(\theta_i) & \cos(\theta_i)\cos(\alpha_{i,i+1}) & -\cos(\theta_i)sin(\alpha_{i,i+1}) & a_{i,i+1}sin(\theta_i) \\ 0 & sin(\alpha_{i,i+1}) & \cos(\alpha_{i,i+1}) & s_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denavit and Hartenberg also use a convenient shorthand notation

$$\begin{vmatrix} a_{i,i+1} \\ \alpha_{i,i+1} \\ \theta_i \\ s_i \end{vmatrix}$$

Another Version of D-H Parametrization

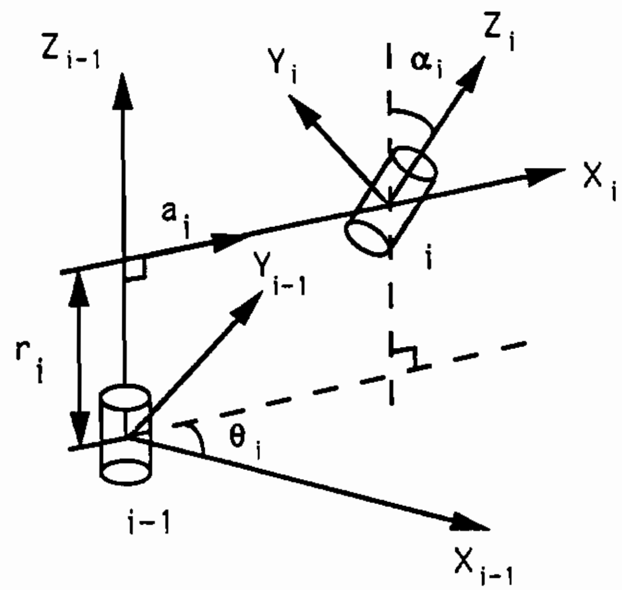


Fig. 1.4.2

a_i= length of common normal between Z_{i-1} and Z_i

α_i= angle between Z_{i-1} and Z_i positive in the right hand sense about the common normal X_i

θ_i = joint angle positive in the right hand sense about Z_i

r_i = length along Z_{i-1} from X_{i-1} to X_i

Consider a system of mechanical links (numbered 1 through N) each capable of linear or rotary motion relative to the adjacent links. For reference, define link 0 to be fixed to a base (on a rigid table, on a vehicle or in the ground). Define a joint to be the intersection of two adjacent links. In particular, **joint i** is the intersection of **links** $i-1$ and i. $i = 1, 2, \cdots N$.

$\vec{Z}_i$ = axis of joint $(i+1)$ (sense arbitrary). If the joint i is an R-pair then $\vec{Z}_i$ is the axis of the rotation; if it is a P pair, then it is the direction of linear motion; if it is an H pair, then it is the screw axis. Defining $\vec{Z}_N$ in an arbitrary manner $\vec{Z}_i$ is defined for $i = 0, 1, 2 \cdots N$.

The vector, $\vec{X}_i = \frac{\vec{Z}_{i-1} \times \vec{Z}_i}{||Z_{i-1} \times Z_i||}$ defines the common normal between $\vec{Z}_{i-1}$ and $\vec{Z}_i$ directed from the former to the latter. If Z_{i-1} and Z_i are such that $\vec{Z}_{i-1} \times \vec{Z}_i = 0$ then $\vec{X}_i$ is arbitrary subject only to the condition $\vec{X}_i \perp \vec{Z}_i$.

Defining $\vec{Z}_o$ in an arbritrary manner we have $\vec{X}_i$ defined for $i = 0, 1, 2, \ldots N$.. Define $\vec{Y}_i = \vec{Z}_i \times \vec{X}_i$, $i = 0, 1, 2, \cdots, N$.

Thus we have defined an orthonormal coordinate system $(\vec{X}_i, \vec{Y}_i, \vec{Z}_i)$ fixed in link i, $i = 0, 1, 2, .., N$. We can state unambiguously,

$\theta_i =$ angle from $\vec{X}_{i-1}$ to $\vec{X}_i$ measured positively counterclockwise about $\vec{Z}_{i-1}$

$r_i =$ distance along $\vec{Z}_{i-1}$ from $\vec{X}_{i-1}$ to $\vec{X}_i$

$\alpha_i =$ angle from $\vec{Z}_{i-1}$ to $\vec{Z}_i$ measured positively counterclockwise about $\vec{X}_i$

$a_i =$ distance along $\vec{X}_i$ from $\vec{Z}_{i-1}$ to $\vec{Z}_i$

A vector $\vec{V}_i$ fixed in coordinate system i can be expressed in system $(i-1)$ as $\vec{V}_{i-1}$ using the transformation T^i_{i-1} as follows:

$$\vec{V}_{i-1} = T^i_{i-1} \vec{V}_i$$

More specifically $\vec{V}_i$ or the frame i is first rotated about $\vec{X}_i$ by $-\alpha_i$ (aligning $\vec{Z}_i$ and $\vec{Z}_{i-1}$ then translated along $\vec{X}_i$ by a_i (bringing $\vec{Z}_i$ and $\vec{Z}_{i-1}$ into coincidence). then translated along $\vec{Z}_i (\equiv \vec{Z}_{i-1})$ by r_i (giving $\vec{X}_i$ and $\vec{X}_{i-1}$ common origin) and finally rotated about $\vec{Z}_i$ by $-\theta_i$ (bringing the two systems into coincidence).

Thus,

$$T^i_{i-1} = \begin{pmatrix} C_{\theta_i} & -S_{\theta_i} & 0 & 0 \\ S_{\theta_i} & C_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{\alpha_i} & -S_{\alpha_i} & 0 \\ 0 & S_{\alpha_i} & C_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where

$$\begin{aligned} C_{\theta_i} &= cos(\theta_i) \\ S_{\theta_i} &= sin(\theta_i) \\ C_{\alpha_i} &= cos(\alpha_i) \\ S_{\alpha_i} &= sin(\alpha_i). \end{aligned}$$

Thus the Denavit-Hartenberg matrix for the joint i is given by

$$T_{i-1}^{i} = \begin{bmatrix} C_{\theta_i} & -C_{\alpha_i}S_{\theta_i} & S_{\alpha_i}S_{\theta_i} & a_iC_{\theta_i} \\ S_{\theta_i} & C_{\alpha_i}C_{\theta_i} & -S_{\alpha_i}C_{\theta_i} & a_iS_{\theta_i} \\ 0 & S_{\alpha_i} & C_{\alpha_i} & r_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here a_i and α_i are constants. If i is an R-pair, then r_i is a constant and θ_i is a variable. If i is a P-pair, then θ_i is a constant and r_i is a variable. If i is a H-pair, then θ_i and r_i are both variables but

$$\delta r_i = h\delta\theta_i, \qquad h = pitch$$

In the next few pages we sketch out the JPL-Stanford Arm (handouts supplied by JPL). The D-H parameters are also given.

Recall that the end-effector coordinate system is $(\vec{X}_N, \vec{Y}_N, \vec{Z}_N)$. In the case of the Stanford-JPL Arm it is the frame $(\vec{X}_6, \vec{Y}_6, \vec{Z}_6,)$ see Fig. 1.4.3 and Fig. 1.4.4. Referring to Figure 1.4.6. the vectors $\vec{n}$, $\vec{s}$, and $\vec{a}$ completely determine the orientation of the end-effector with respect to the base $\equiv (\vec{X}_o, \vec{Y}_o, \vec{Z}_o)$ frame. The origin of the end-effector frame is given by a vector $\vec{p}$. Now any four vector of the form $(x, y, z, 1)$ in the end-effector coordinate system, has coordinates in the base system given by

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{base} = (T_0^1 \cdot T_1^2 \ldots T_{N-1}^N) \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{end-effector}$$

$$= T_0^N \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{end-effector}$$

Now consider separately the cases in which the vector in end-effector coordinates is taken to be $V_1 = (0,0,0,1)', V_2 = (1,0,0,1)', V_3 = (0,1,0,1)'$ and $V_4 = (0,0,1,1)'$

Now V_1 is the origin of the $(\vec{X}_N, \vec{Y}_N, \vec{Z}_N)$ coordinate system and hence is given by the tip of the vector $\vec{p}$. Thus,

$$T_0^N\ V_1 = (V_1)_{base} = \vec{p}$$

The vectors V_2, V_3 and V_4 are respectively the tips of the vectors $\vec{n}$. $\vec{s}$ and $\vec{a}$. But the latter are given in base coordinates by $(\vec{n} + \vec{p})$. , $(\vec{s} + \vec{p})$ and $(\vec{a} + \vec{p})$.

On the other hand,

REFERENCE FRAMES FOR LINK-JOINT PAIRS OF

JPL - STANFORD ARM

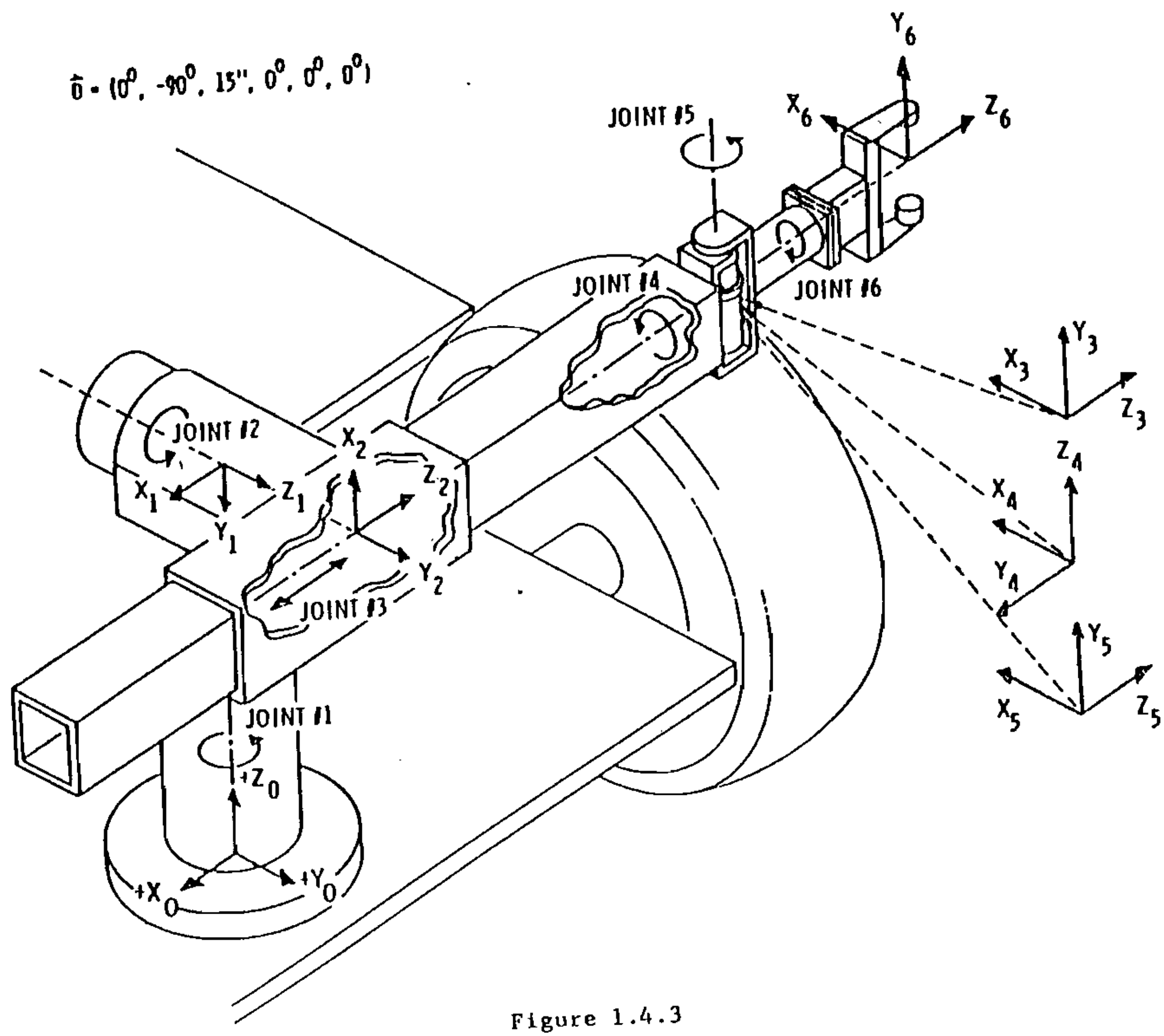


Figure 1.4.3

JPL - STANFORD ARM

Joint Variables θ_1, θ_2, r_3, θ_4, θ_5, θ_6, and Constant Orthogonal Displacements r_1, r_2, r_6

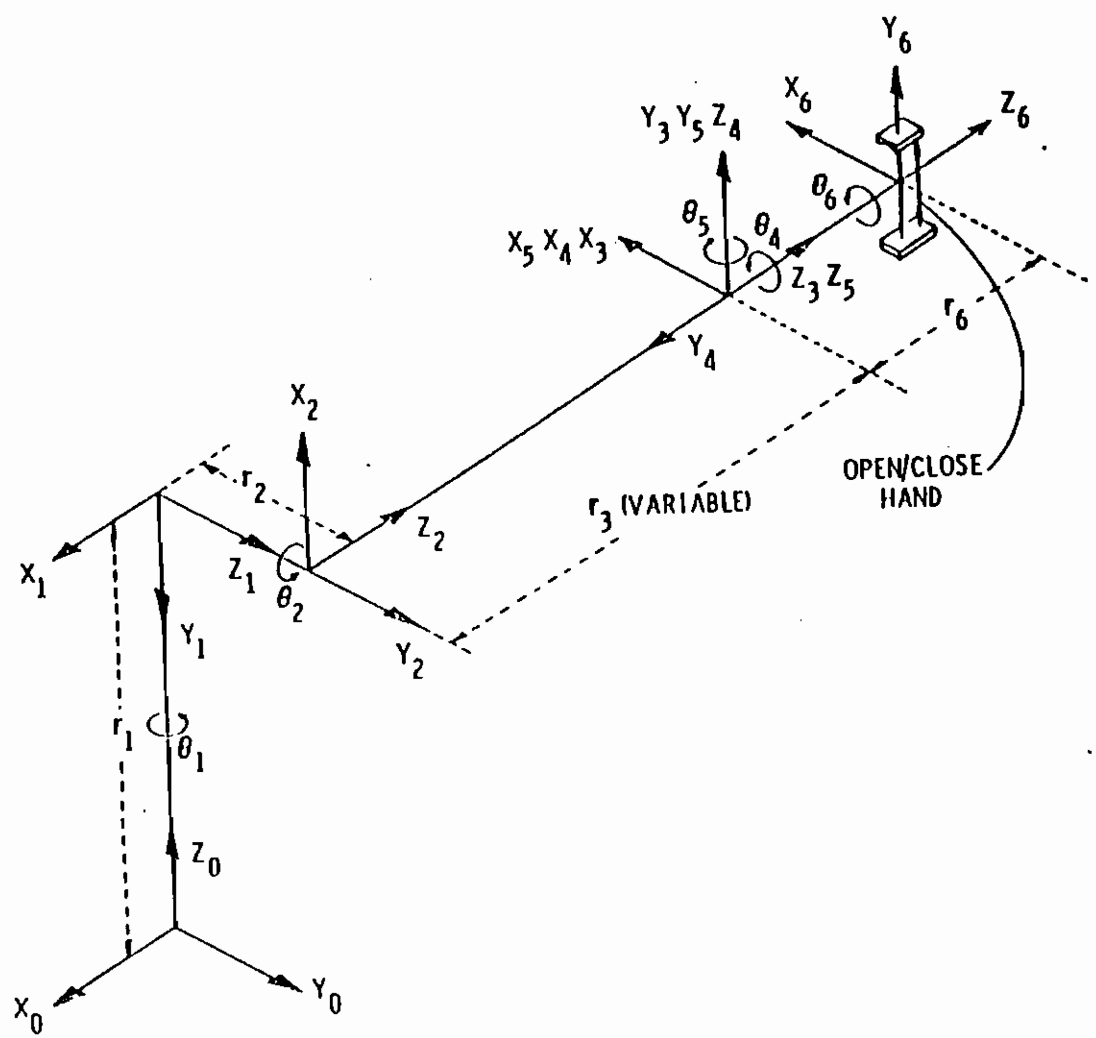


Figure 1.4.4

JPL - STANFORD ARM

Joint Variables θ_1, θ_2, r_3, θ_4, θ_5, θ_6, and Constant Orthogonal Displacements r_1, r_2, r_6

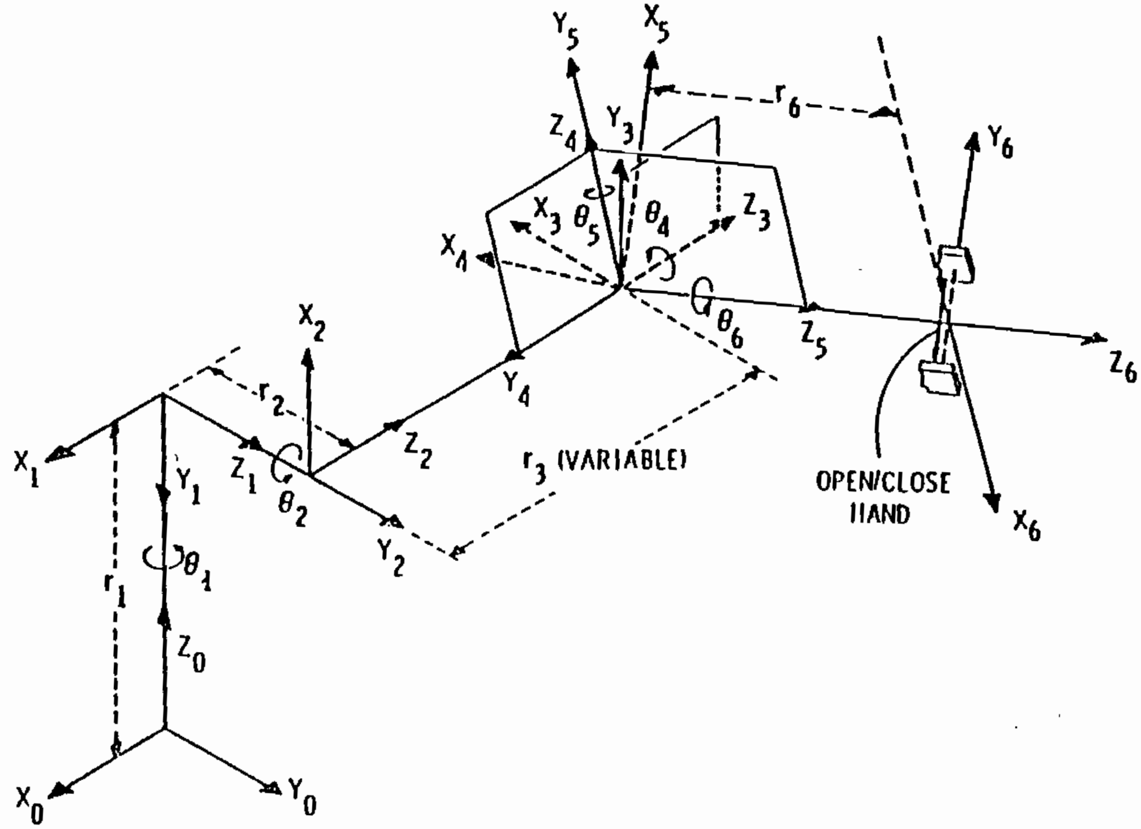
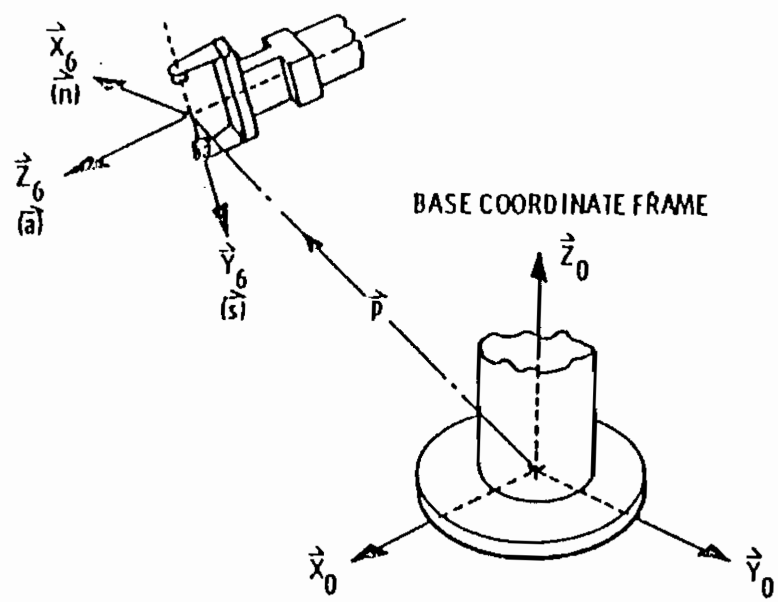


Figure 1.4.5

JPL - STANFORD ARM

HAND POSITION AND ORIENTATION EXPRESSED IN BASE (OR WORLD) COORDINATES



$$\vec{\theta} \equiv (\theta_1, \theta_2, r_3, \theta_4, \theta_5, \theta_6).$$

Figure 1.4.6

$$
\begin{aligned}
(\vec{n} + \vec{p})_{base} = (V_2)_{base} &= T_0^N (V_2) \\
&= T_0^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= T_0^N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + T_0^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \vec{p} + T_0^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Thus $(\vec{n})_{base} = T_0^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and similarly for $\vec{s}$ and $\vec{a}$.

Thus we see that

$$
T_0^N = \left[\begin{array}{c|c} A & p \\ \hline 0 & 1 \end{array} \right]
$$

when $A = (n, s, a)$ is the matrix of direction cosines of the vectors $\vec{n}$, $\vec{s}$, $\vec{a}$ with respect to the base coordinate system $(\vec{X}_o,\ \vec{Y}_o,\ \vec{Z}_o)$

From now on, the end-effector as in Fig. 1.4.6 will be called a **hand**. More sophisticated hands will be of interest to us later.

We have shown that the hand position (p) and orientation A can be determined in terms of the joint variables:

$$
\begin{aligned}
\begin{bmatrix} A & p \\ 0 & 1 \end{bmatrix} &= T_0^N \\
&= T_0^1 T_1^2 \dots T_{N-1}^N
\end{aligned}
\tag{1.4.1}
$$

Now the fundamental problems of robotic manipulation are of the following form:

Given an initial condition $\begin{pmatrix} A_0 & p_0 \\ 0 & 1 \end{pmatrix}$ for the hand it is desired to find time functions for the **joint variables** which will ensure that at time t_1 the final condition $\begin{pmatrix} A_1 & p_1 \\ 0 & 1 \end{pmatrix}$ will be attained. This is a low level control problem and we shall see how to systematically attack problems of this type.

Once the right hand side of 1.4.1 is written explicity in terms of the joint variables we get a set of equations for the position and orientation of the manipulator end-effector.

These are what we call the **kinematic equations** of the manipulator. For example. the Standford-JPL arm has 6 degrees of freedom and is a linkage of the type $RRPRRR$.

Table 1.4.3 gives an explicit form of the kinematic equations for the JPL-Stanford arm.

A key observation that deserves to be made is the following. Associated to an R pair i, an angle θ_i enters the kinematics equation. Only expressions of the form $cos\theta_i$ and $\sin\theta_i$ enter the kinematic equations.

This suggests we define

$$x_i = cos(\theta_i)$$
$$y_i = sin(\theta_i) \tag{1.4.2}$$

and add the constraint

$$x_i^2 + y_i^2 = 1 \tag{1.4.3}$$

If we do this for every R pair then the kinematic equations become a system of **n algebraic equations** in **m variables** where,

$$\begin{aligned} m &= 2 \times N_R + (N - N_R) \\ &= N + N_R \end{aligned}$$

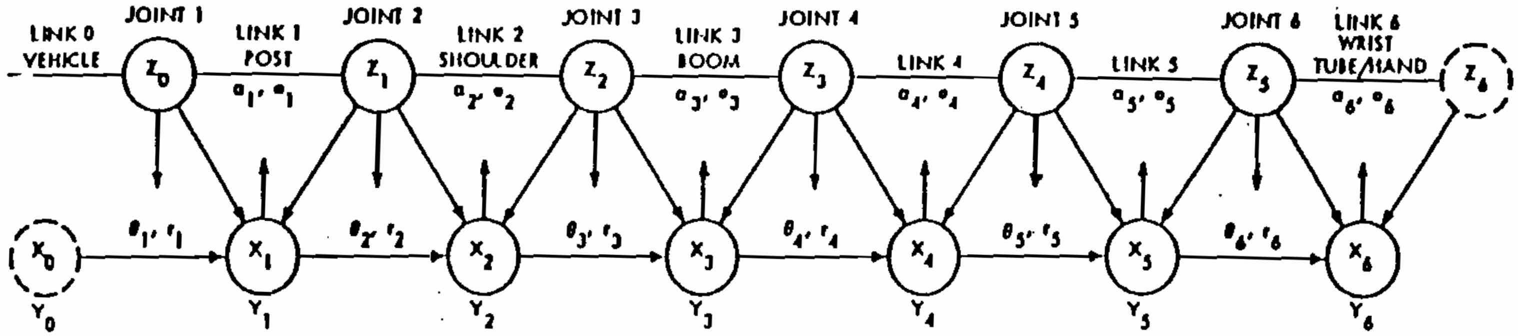
where, N_R = number of R-pairs. The number of equations

$$n = 6 + 3 + N_R$$

The key point here is that we can try to use the tools of **algebraic geometry** to understand the structure of the set of solutions to the system of equations so obtained. Of

AN EXAMPLE, THE JPL-STANFORD ARM,

USING THE HARTENBERG - DENAVIT REPRESENTATION



i	α_i (deg)	r_i (in.)	θ_i (deg)	a_i (in.)	Maximum radial dimension about z_i (in.)	Maximum linear extension along $-z_i$ (in.)
1	-90	14	[-175, 175]	0	2.625	0
2	90	6.375	[-175, 175]	0	2.75	12
3	0	[5.5, 44]	-90	0	1.25	$55-r_3$
4	-90	0	[-175, 175]	0	2.375	0
5	90	0	[-110, 110]	0		0
6	0	9.75	[-175, 175]	0		0
fingers	5 in. long, 1 3/16 in. wide, open to 2 1/4 in. each					

Note: The ranges of the six joint variables are indicated in brackets above.

Table 1.4.1

Table 1.4.2

JPL–STANFORD ARM

Individual Link Transformation

$$T_0^1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & r_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_1^2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & r_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_3^4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_5^6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & r_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The arm transformation T is given by

$$T \equiv T_0^6 = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6.$$

Table 1.4.3

Arm Transformation

$$T = T_0^6 = \begin{bmatrix} \vec{n} & \vec{s} & \vec{a} & \vec{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} c_{1256} s_4 + s_1 c_{456} - c_{16} s_{25} + c_{124} s_6 - s_{146} \\ s_{14} c_{256} - c_{1456} - s_{125} c_6 + s_{16} c_{24} + c_1 s_{46} \\ -s_{24} c_{56} - c_{26} s_5 - s_{26} c_4 \end{bmatrix}$$

$$\vec{s} = \begin{bmatrix} -c_{125} s_{46} - s_{16} c_{45} + c_1 s_{256} + c_{1246} - s_{14} c_6 \\ -s_{146} c_{25} + c_{145} s_6 + s_{1256} + s_1 c_{246} + c_{16} s_4 \\ s_{246} c_5 + c_2 s_{56} - s_2 c_{46} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} c_{12} s_{45} + s_{15} c_4 + c_{15} s_2 \\ s_{145} c_2 - c_{14} s_5 + s_{12} c_5 \\ c_{25} - s_{245} \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} r_3 c_1 s_2 - r_2 s_1 \\ r_3 s_1 s_2 + r_2 c_1 \\ r_3 c_2 + r_1 \end{bmatrix} + r_6 \hat{a}$$

NOTE: $s_i \equiv \sin\theta_i \quad c_i \equiv \cos\theta_i$

$s_{ij} \equiv \sin\theta_i \sin\theta_j$

$c_{ijk} = \cos\theta_i \cos\theta_j \cos\theta_k$, etc.

particular interest is the possibility that when the manipulator is redundant ($N > 6$) such a system of equation may have multiple solutions of which certain solutions may be preferable to certain other solutions such as when we take into account work-space constraints and the need for collision avoidance.

Before we close this section we would like to encourage the reader to try his/her hand at working out some of the kinematic equations in the above- mentioned algebraic form for a specific manipulator such as the JPL- Stanford Arm. A second **related exercise** is to prove the following,

Theorem (Peiper): If a given minipulator has 6 degrees of freedom and if the fourth, fifth and sixth joints are R-pairs with axes intersecting at a common point, then the problem of solving the kinematic equations **decouples** into two **independent** subproblems:

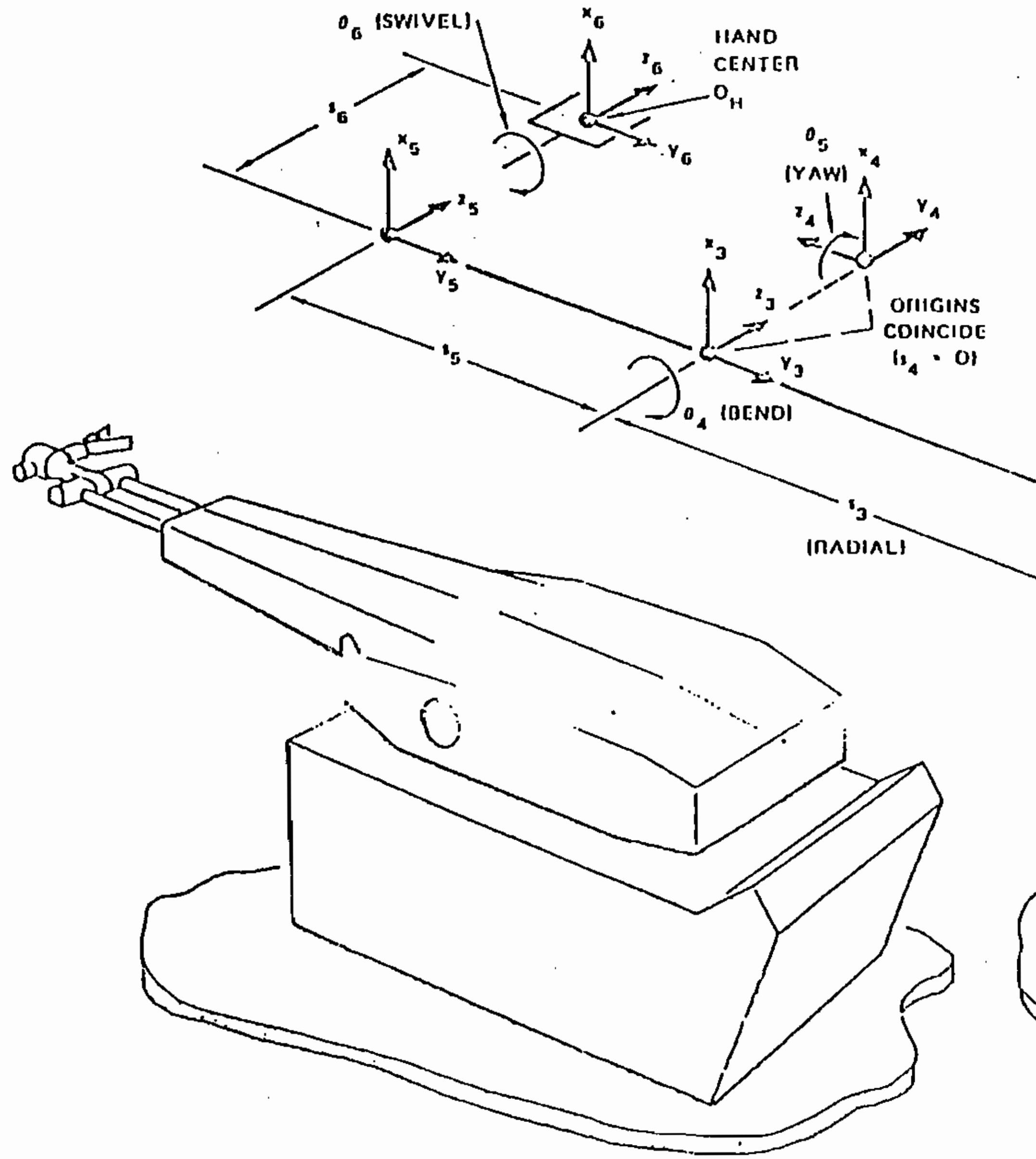
(1) the problem of computing joint angles to fix the orientation of the end-effector.

(2) the problem of computing the joint angles for the joints 1,2,3 to fix the position of the origin of the end-effector.

Remark: A manipulator in which the last three joints are cointersecting R-pairs is said to have a spherical wrist.

ANOTHER EXAMPLE FOR H-D MATRIX REPRESENTATION

(UNIMATE ROBOT ARM)



JOINT COORDINATES FOR POSITION ILLUSTRATED		
θ_1	-90°	ROTARY
θ_2	-90°	VERTICAL
s_3	—	RADIAL
θ_4	0°	BEND
θ_5	0°	YAW
θ_6	0°	SWIVEL

Figure 1.4.7