1.5 The Forward Kinematics and Jacobians

Let q_i denote the i^{th} ***generalized coordinate*** or configuration variable of an N link manipulator. Thus,

$$q_i = \begin{cases} \theta_i & \text{if } i^{th} \text{ joint is a R pair,} \\ r_i & \text{if } i^{th} \text{ joint is a P pair.} \end{cases}$$

Let $q = (q_1, \cdots, q_N)$ define the n tuple of generalized coordinates. The matrix T_0^N defined in Section 1.4 depends on q. As q varies, T_0^N takes different values in the Euclidean group. The ***forward kinematic map*** is the map

$$\begin{aligned} F : Q &\longrightarrow SE(3) \\ q &\mapsto T_0^N\,(q). \end{aligned}$$

Here Q denotes the configuration space. For a manipulator with all joints revolute, $Q = S^1 \times \cdots \times S^1$, the N-torus.

In section 1.4 we have shown that the forward kinematic map of an N-link *serial* manipulator is simply given by a product of matrices all of which are exponentials.

In operation of a robot, the computer control system determines a curve $q(t)$ as a function of time, which in turn determines a curve $F(q(t))$ in the Euclidean group. It is of interest to describe such curves in terms of infinitesimal objects e.g. velocities. This leads to the concept of Jacobian. To understand this, it is worthwhile to consider the case of $N = 1$, i.e. a simple rigid body.

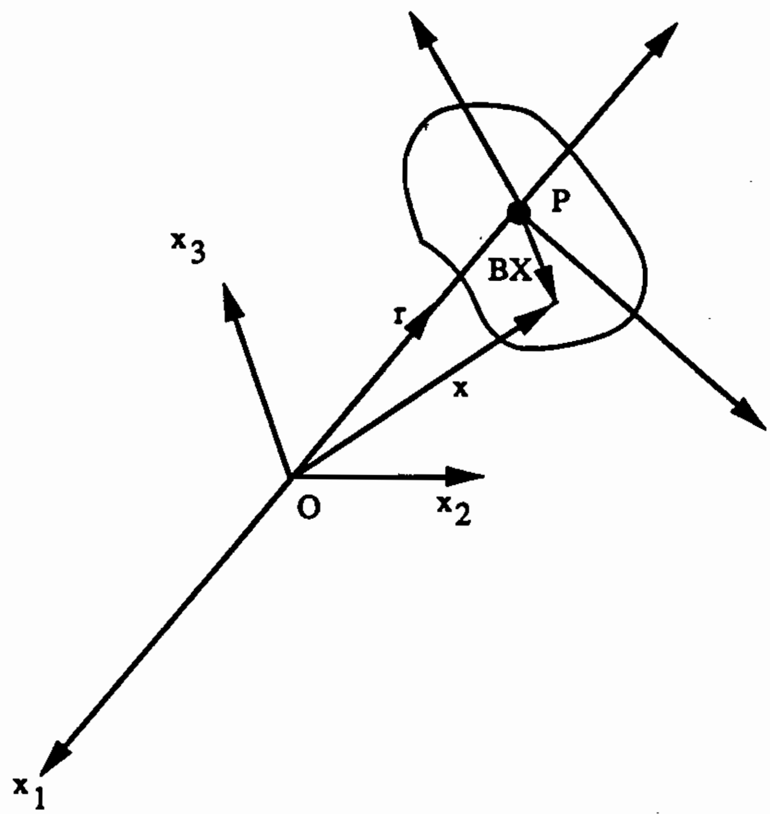

Figure 1.5.1 Rigid Body Kinematics

Referring to Figure 1.5.1, we note that

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} B & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

and thus,

$$T_0^1 \triangleq \begin{pmatrix} B & r \\ 0 & 1 \end{pmatrix} \in SE(3).$$

Leaving aside for the moment the specific nature of the joint (if any) connecting the rigid body to the laboratory frame, let

$$t \longmapsto \begin{pmatrix} B(t) & r(t) \\ 0 & 1 \end{pmatrix}$$

denote a differentiable function with values in $SE(3)$, - a *motion* of the rigid body.

Since, $B^T(t)\, B(t) = \mathbf{1} \quad \forall t$, differentiating, we get

$$\dot{B}^T B + B^T \dot{B} = 0.$$

Equivalently,

$$(B^T \dot{B})^T + B^T \dot{B} = 0.$$

Thus $B^T \dot{B}$ is a 3×3 skew-symmetric matrix which we denote as $\hat{\Omega}$. Equivalently, $\dot{B} = B\hat{\Omega}$.

Here $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T \in \mathbb{R}^3$ denotes the ***body-angular-velocity*** of the body.

It is easy to see that, if we define, $\omega = B\Omega$ then,

$$\dot{B} = \hat{\omega}\, B$$

(Hint: Use $\widehat{B\Omega} = B\hat{\Omega}\, B^T$).

This vector ω is called the ***spatial angular velocity*** of the rigid body.

We can write, for velocity and acceleration,

$$\begin{aligned}
\dot{x} &= \frac{d}{dt}(BX + r) \\
&= B\hat{\Omega}X + B\dot{X} + \dot{r} \\
&= B(\Omega \times X + \dot{X}) + \dot{r} \\
\ddot{x} &= \dot{B}(\hat{\Omega}\, X + \dot{X}) + B(\hat{\Omega}\, X + \hat{\Omega}\dot{X} + \ddot{X}) + \ddot{r} \\
&= B(\hat{\Omega}^2\, X + 2\hat{\Omega}\dot{X} + \hat{\Omega}X + \ddot{X}) + \ddot{r}
\end{aligned}$$

The term $2\hat{\Omega}\dot{X}$ is the *Coriolis* term, and $\hat{\Omega}^2\, X$ is the *centrifugal* term. In the case when the material point of interest in the body is fixed in the body, i.e. $\dot{X} = 0$, we can write

$$\begin{aligned}\begin{pmatrix}\dot{x}\\0\end{pmatrix} &= \begin{pmatrix}\dot{B} & \dot{r}\\0 & 0\end{pmatrix}\begin{pmatrix}X\\1\end{pmatrix}\\ &= \begin{pmatrix}\dot{B} & \dot{r}\\0 & 0\end{pmatrix}\begin{pmatrix}B & r\\0 & 1\end{pmatrix}^{-1}\begin{pmatrix}x\\1\end{pmatrix}\\ &= \begin{pmatrix}\dot{B}B^T & \dot{r} - \dot{B}B^T r\\0 & 0\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix}\\ &= \begin{pmatrix}\hat{\omega} & \dot{r} - \hat{\omega}r\\0 & 0\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix}.\end{aligned}$$

We will call the vector in $\mathbb{R}^6$,

$$\begin{pmatrix}\omega\\ \dot{r} - \omega \times r\end{pmatrix}$$

the instantaneous *twist velocity* of the body. It is clear that the twist velocity contains the infinitesimal motion information. The key idea behind the Jacobian is that it gives a formula that relates the twist velocity to *joint rates* $\dot{q}_i$ etc.

Consider a general N link manipulator with forward kinematic map $q \mapsto F(q) = T_0^N(q)$,

$$T_0^N(q) = T_0^1(q_1)\, T_1^2(q_2) \cdots T_{i-1}^i(q_i) \cdots T_{N-1}^N(q_N),$$

where each $T_{i-1}^i(q_i)$ is a 4×4 Denavit - Hartenberg matrix.

When the joints are set in motion, $q_i = q_i(t)$ a function of time,

$$\frac{dT_0^N}{dt} = \sum_{i=1}^{N} \frac{\partial T_0^N}{\partial q_i}\, \dot{q}_i,$$

where,

$$\begin{aligned}
\frac{\partial T_0^N}{\partial q_i} &= T_0^1\,(q_1)\,T_1^2\,(q_2)\cdots T_{i-2}^{i-1}\,(q_{i-1})\,\frac{\partial T_{i-1}^i}{\partial q_i}\,T_i^{i+1}\cdots T_{N-1}^N \\
&= T_0^1(q_1)\,T_1^2(q_2)\cdots T_{i-2}^{i-1}\,(q_{i-1})\,M_i\,T_{i-1}^i(q_i)\cdots T_{N-1}^N,
\end{aligned}$$

$$M_i = \begin{cases} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{if } i^{th} \text{ joint is revolute,} \\[2ex] \left[\begin{array}{ccc|c} & & & 0 \\ & \bigcirc & & 0 \\ & & & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] & \text{if } i^{th} \text{ joint is prismatic.} \end{cases}$$

$$\begin{aligned}
\Rightarrow \frac{\partial T_0^N}{\partial q_i} &= (T_0^1\,T_1^2\cdots T_{i-2}^{i-1})\,M_i\,(T_0^1\,T_1^2\cdots T_{i-2}^{i-1})^{-1}\,T_0^N \\
&= \Delta_i\,T_0^N.
\end{aligned}$$

(By convention $T_0^0 = \mathbf{1}_4$ the identity.)

It follows that,

$$\frac{d}{dt}T_0^N = \left(\sum_{i=1}^{N} \Delta_i\,\dot{q}_i\right) T_0^N,$$

or equivalently

$$\dot{T}_0^N\,(T_0^N)^{-1} = \sum_{i=1}^{N} \Delta_i \dot{q}_i.$$

Denote,

$$\left[\begin{array}{c|c} \hat{\omega}_0^N & \dot{r}_0^N - \hat{\omega}_0^N\,r_0^N \\ \hline 0 & 0 \end{array}\right] = \dot{T}_0^N\,(T_0^N)^{-1}.$$

Thus,

$$\begin{bmatrix} \omega_0^N \\ \dot{r}_0^N - \omega_0^N \times r_0^N \end{bmatrix}$$

is the twist velocity of the hand frame.

We have shown that

$$\begin{bmatrix} \omega_0^N \\ \dot{r}_0^N - \omega_0^N \times r_0^N \end{bmatrix} = \left[\begin{array}{c|c|c|c} \delta_1 & \delta_2 & \cdots & \delta_N \\ \eta_1 & \eta_2 & & \eta_N \end{array}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_N \end{bmatrix}$$

where $\delta_i,\ \eta_i$ are such that

$$\Delta_i = \left[\begin{array}{c|c} \hat{\delta}_i & \eta_i \\ \hline 0 & 0 \end{array}\right].$$

One can write recursive formulas for the $\delta_i,\ \eta_i$. Recall that

$$\Delta_i = T_0^1\ T_1^2\ \cdots\ T_{i-2}^{i-1}\ M_i (T_0^1\ T_1^2\ \cdots\ T_{i-2}^{i-1})^{-1}.$$

Define

$$\left[\begin{array}{c|c} X_i & \xi_i \\ \hline 0 & 1 \end{array}\right] \triangleq T_0^{i-1} = T_0^1\ T_1^2\ \cdots\ T_{i-2}^{i-1}.$$

By convention,

$$X_1 = \mathbf{1},$$
$$\xi_1 = 0.$$

Then, if the i^{th} joint is revolute,

$$\Delta_i = \left[\begin{array}{c|c} X_i & \xi_i \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left[\begin{array}{c|c} X_i^T & -X_i^T\ \xi_i \\ \hline 0 & 1 \end{array}\right]$$
$$= \left[\begin{array}{c|c} \widehat{X_i^{(3)}} & -X_i^{(3)} \times \xi_i \\ \hline 0 & 0 \end{array}\right]$$

where $X_i^{(3)}$ is the 3^{rd} column of X_i. If the i^{th} joint is prismatic then,

$$\Delta_i = \left[\begin{array}{c|c} X_1 & \xi_i \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \bigcirc & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} X_i^T & -X_i^T \, \xi_i \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \bigcirc & X_i^{(3)} \\ \hline 0 & 0 \end{array}\right].$$

It follows that,

$$\begin{pmatrix} \delta_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} X_i^{(3)} \\ -X_i^{(3)} \times \xi_i \end{pmatrix} \text{ if } i^{th} \text{ joint is revolute}$$

$$= \begin{pmatrix} 0 \\ --- \\ X_i^{(3)} \end{pmatrix} \text{ if } i^{th} \text{ joint is prismatic.}$$

It is easy to calculate the $X_i^{(3)}$, ξ_i etc recursively.

$$\text{Let } T_{i-1}^i = \left[\begin{array}{c|c} A_i & b_i \\ \hline 0 & 1 \end{array}\right].$$

Then, with initial conditions, $X_1 = \mathbf{1}$ and $\xi_1 = 0$,

$$\left.\begin{aligned} X_i &= X_{i-1} \, A_{i-1} \\ \xi_i &= \xi_{i-1} + X_{i-1} \, b_{i-1} \end{aligned}\right\} \quad i = 2, \cdots, N$$

$$\begin{pmatrix} \delta_i \\ --- \\ \eta_i \end{pmatrix} = \begin{cases} \begin{bmatrix} X_i^{(3)} \\ -X_i^{(3)} \times \xi_i \end{bmatrix} & \text{if } i^{th} \text{ joint is revolute} \\ \begin{bmatrix} 0 \\ X_i^{(3)} \end{bmatrix} & \text{if } i^{th} \text{ joint is prismatic} \end{cases}$$

and

$$\mathbb{J} = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_N \\ \eta_1 & \eta_2 & \cdots & \eta_N \end{bmatrix} \text{ is the Jacobian.}$$

Very often one is interested in the twist velocity of the end-effector relative to frame other then the customary base frame or lab frame. For instance, if the gripper is used in robotic assembly then the twist velocity relative to the *assembly fixture* is needed so as to avoid collisions and insure task completion. Denote the fixture frame as $(X_f,\ Y_f,\ Z_f)$ as in the Figure 1.5.2.

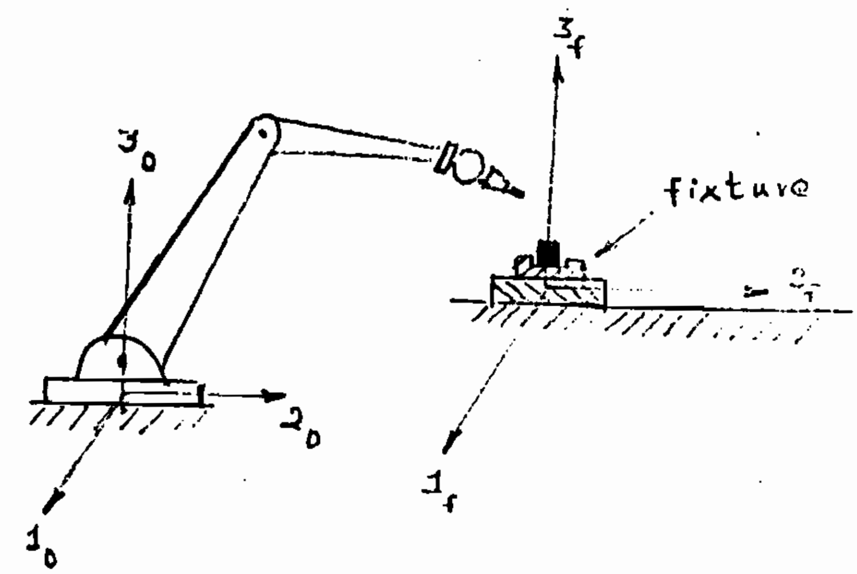

Figure 1.5.2 Fixture Frame

Let T_f^0 denote the Euclidean group element that determines the base frame relative to the fixture frame. Let

$$\begin{pmatrix} \omega \\ v \end{pmatrix} = \begin{pmatrix} \omega_0^N \\ \dot{r}_0^N - \omega_0^N \times r_0^N \end{pmatrix}$$

denote the twist velocity of the end-effector in the base frame $(X_0,\ Y_0,\ Z_0)$. Let $\begin{pmatrix} \tilde{\omega} \\ \tilde{v} \end{pmatrix}$ denote the twist velocity of the end-effector in the fixture frame. Suppose

$$T_f^0 \triangleq \left[\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right].$$

Then,

Lemma $\begin{pmatrix} \tilde{\omega} \\ \tilde{v} \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ --- & --- \\ \hat{b}A & A \end{array}\right) \begin{pmatrix} \omega \\ v \end{pmatrix}$

Proof $\left(\begin{array}{c|c} \hat{\tilde{\omega}} & \tilde{v} \\ 0 & 0 \end{array}\right) = \dot{T}_f^N \left(T_f^N\right)^{-1}$

$$
\begin{aligned}
&= (T_f^0 \, \dot{T}_0^N) \; (T_f^0 \, T_0^N)^{-1} \\
&= \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \dot{T}_0^N \, T_0^{N-1} \begin{pmatrix} A^T & -A^T b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \left(\begin{array}{c|c} \hat{\omega} & v \\ -- & -- \\ 0 & 0 \end{array}\right) \begin{pmatrix} A^T & -A^T b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} A\hat{\omega}A^T & Av - A\hat{\omega}A^T b \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \widehat{A\omega} & Av - \widehat{A\omega} b \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \widehat{A\omega} & Av + \hat{b}A\omega \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \begin{pmatrix} \tilde{\omega} \\ \tilde{v} \end{pmatrix} &= \begin{pmatrix} A\omega \\ Av + \hat{b}A\omega \end{pmatrix} \\
&= \left(\begin{array}{c|c} A & 0 \\ -- & -- \\ \hat{b}A & A \end{array}\right) \begin{pmatrix} \omega \\ v \end{pmatrix}
\end{aligned}
$$

Remark Observe that the transformation rule is identical to that for Plücker coordinates of a line. This is not accidental.