2. Dynamics

We begin by discussing briefly, the dynamics of particles and rigid bodies from the balance laws of Newton and Euler to the variational principle of Langrange and Hamilton. The latter can be used effectively to quickly generate equations of motion for a manipulator composed of linked rigid bodies.

2.1 **Particles**

From Newton, for a particle of mass m, coordinates x (in the laboratory/ inertial frame).

$$f = m\ddot{x} \tag{2.1.1}$$

Additionally, if we let $p = m\dot{x}$ and $\ell = xxp$ denote the augular momentum,

$$\begin{aligned} \dot{\ell} &= \dot{x}\ x\ p + x\ x\ \dot{p} \\ &= \frac{p}{m}\ x\ p + x\ x\ f \\ &= x\ x\ f. \end{aligned} \tag{2.1.2}$$

The quantity $x\ x\ f$ = torque on the particle due to the force f. Equation 2.1.2 is referred to as Euler's balance law, and for particles it is clearly a **derived** principle, not a fundamental one, since it is a consequence of (2.1.1). To work out the equations of motion of a rigid body, one could proceed along similar lines, this time modeling also the internal forces of the rigid body that hold the particles in the rigid body together. The necessary hypothesis on internal forces is that they are along the lines joining the particles and directed opposite to each other but equal in magnitude. Then one computes the force resultant and applies (2.1.1) and (2.1.2) to the entire system of particles.

This is unsatisfactory, in that the hypothesis on internal forces is not (easily) checkable. This is overcome by following Euler and treating the rigid body as a fundamental unit (idealization). Newton's and Euler's balance laws are then **axioms**.

2.2 Rigid Bodies

Consider a rigid body as in the figure 2.1,

Figure 2.1 Rigid Body

The body is subject top a pure force f along a specified line of action and a pure couple c_0. A particular point (but arbitrary), is determined on the line of action of f by the vector γ. Assume that the frame of reference on the body is attached at its origin 0 to the center of the mass of the rigid body. We note that this data will be used to compute the resultant couple c on the body.

From Newton,

$$f = m\ddot{r} \tag{2.2.1}$$

From Euler,

$$c = \dot{\ell} \tag{2.2.2}$$

What are ℓ and c in this setting?

The total angular momentum of the rigid body,

$$
\begin{aligned}
\ell &= \int_\beta x\ x\ \dot{x}\ \ dm(x) \\
&= \int_\beta (BX\ +\ r)\ x\ (\dot{B}X\ +\ \dot{r})\ dm(X) \\
&= (BX\ +\ r)\ x\ (B\hat{\Omega}X\ +\ \dot{r})\ dm(X) \\
&= B\ \int_\beta X\ x\ (\Omega\ \times\ X)\ dm(X) \\
&\qquad +\ m\ rx\dot{r}.
\end{aligned} \tag{2.2.3}
$$

Here we have used the fact that

(i) $(BU)\ \times\ (BW)\ =\ B(V\ \times\ w)$ for $B \epsilon\ S0(3)$;

(ii) $\int_\beta\ X\ dm(X)\ =\ 0$,

since the center of mass of β coincides with the origin of the body frame.

Now

$$
\begin{aligned}
X\ \times\ (\Omega\ \times\ X)\ &=\ (X\cdot X)\Omega \\
&\quad -\ (X\cdot\Omega)X \\
&=\ (||X||^2\mathbb{1}\ -\ XX^T)\Omega.
\end{aligned}
$$

Define $\mathbb{I}\ =\ \int_\beta (||\ X\ ||^2\ \mathbb{1}\ -\ XX^T)\ dm\ (X)$, to be the moment of inertia of the body in the body frame. Then, the angular momentum,

$$
\begin{aligned}
\ell &= B\ \mathbb{I}\ \Omega\ +\ m\ r\ x\ \dot{v} \\
&= B\ \mathbb{I}\ \Omega\ +\ r\ x\ p
\end{aligned} \tag{2.2.4}
$$

where $p\ =\ m\dot{r}$ is the linear momentum. From figure 2.1, the torque/ couple resultant is

$$
c\ =\ c_o\ +\ \gamma\ \times\ f \tag{2.2.5}
$$

(Verify that this is independent of the choice of the point γ on the line of action of the force !)

$$\text{Then } \dot{\ell} = \frac{d}{dt}(B\,\mathbb{I}\,\Omega + rxp)$$
$$= c_o + \gamma x f.$$

Since $\dot{B} = B\hat{\Omega}$,

$$\begin{aligned} \dot{\ell} &= B\,\mathbb{I}\,\dot{\Omega} + B\hat{\Omega}\,\mathbb{I}\,\Omega + \dot{r}xp \\ &\qquad\qquad + rx\dot{p} \\ &= B\,\mathbb{I}\,\dot{\Omega} + B(\Omega \times \mathbb{I}\,\Omega) + 0 + rxf \\ &= c_o + \gamma x f \end{aligned}$$

Then,

$$\begin{aligned} B(\mathbb{I}\,\dot{\Omega} + \Omega x\,\mathbb{I}\,\Omega) &= c_o + (\gamma - r)xf \\ &= c_o + dxf. \end{aligned}$$

We thus summarize,

$$f = m\ddot{r} \qquad (a)$$
$$c_o + dxf = B(\mathbb{I}\,\dot{\Omega} + \Omega x\,\mathbb{I}\,\Omega) \qquad (2.2.5)$$
$$(b)$$

Letting $C_o = B^T c_o$ and $D = B^T d$, $F = B^T f$, we can express Euler's Balance low is the more common form,

$$\begin{aligned} \mathbb{I}\,\dot{\Omega} &= \mathbb{I}\,\Omega \times \Omega + C_o + D \times F \\ &= \mathbb{I}\,\Omega \times \Omega + C \end{aligned} \qquad (2.2.7)$$

where C = torque resultant in body frame. If we define the moment of inertia in the laboratory frame to be,

$$\mathbb{I}_{lab} = B\,\mathbb{I}\,B^T,$$

$$\text{then } \beta\,\mathbb{I}\,\Omega = \mathbb{I}_{lab}\,\omega \text{ and hence}$$

$$\beta(\mathbb{I}\,\dot{\Omega} + \Omega \times \mathbb{I}\,\Omega) = \frac{d}{dt}(\mathbb{I}_{lab}\,\omega).$$

Thus, the balance laws of a rigid body are:

$$\begin{bmatrix} f \\ c_o + d \times f \end{bmatrix} = \frac{d}{dt}\left\{\begin{bmatrix} m\mathbf{1} & 0 \\ 0 & \mathbb{I}_{lab} \end{bmatrix} \begin{bmatrix} v_r \\ \omega \end{bmatrix}\right\} \tag{2.2.8}$$

where, $v_r = \dot{r}$. This statement of the balance laws is somewhat in the spirit of expressing the conditions in terms of twists and wrenches. The vector

$$\begin{bmatrix} m\mathbf{1} & 0 \\ 0 & \mathbb{I}_{lab} \end{bmatrix} \begin{bmatrix} v_r \\ \omega \end{bmatrix}$$

plays the part of a "generalized momentam". Note however that $\mathbb{I}_{lab}$ is time-dependent:

2.3 Langrangian Mechanics

In treating the mechanics of particles subject to pure **conservative** forces of the form $f = -\frac{\partial V}{\partial x}$ where $v : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuously differentiable function, the potential energy, one notices (as Lagrange did), that Newton's equations 2.1.1 are equivalent to Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2.3.1}$$

where $L = \frac{1}{2} m \parallel \dot{x} \parallel^2 - v$ is the Lagrangian. Further (2.3.1) continual the first order necessary condition to be satisfied by a continuously differentiable curve $x(\cdot)$ with end points $x(t_1)$ and $x(t_2)$ fixed and minimizing the functional.

$$I = \int_{t_1}^{t_2} L(x(t),\ \dot{x}(t))\, dt \tag{2.3.2}$$

The proof relies on the observation (see D. Luenberger: Optimization by Vector Space Methods, Wiley, 1969, pp 179-180), that the **Gateaux differential** of I on an admissible variation of the given trajectory $x(\cdot)$ is

$$\gamma \mathbb{I}\,(x;\ h) = \int_{t_1}^{t_2} (\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}})\, h(t)\, dt \tag{2.3.3}$$

where admissibility implies $h(t_1) = h(t_2) = 0$ and $\dot{h}$ is continuous, but otherwise $h(\cdot)$ is arbitrary.

For $\mathbb{I}$ to be a minimum (locally), it is necessary that $\partial I(x;\ h) = 0$ for all admissible h. From this (2.3.1) follows.

For a system (or a particle) subject to **additional** external force, the principle of Langrange and D'Alembert says

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = f_{ext} \tag{2.3.4}$$

The principle holds without change for a system if x is now a generalized coordinate and f_{ext} is the corresponding generalized force. By this we mean that **virtual power** $= f_{ext} \cdot \dot{x}$ where the dot product is suitably interpreted.

More precise and mathematically pleasing treatments of the Lagrange D'Alembert principle (on arbitrary manifolds) can be given but this requires additional technical machinery - which we shall skip.

One goal is to use (2.3.4) in derving the equation of motion of a manipulator.

2.4 Dynamics of a Serial Link Manipulator
(all links assumed rigid)

Assume that a suitable conversion (e.g. Denavit-Hartenberg) has been used to attach frames of reference to each link of a manipulator. Let the i^{th} link be as in figure 2.2.

The origin 0 is not assumed to be the center of mass of link (i). The element of the Euclidean group

$$T_0^i = \begin{pmatrix} {}^iB & {}^ir \\ 0 & 1 \end{pmatrix} \tag{2.4.1}$$

determines the instantaneous configuration of link (i) in the laboratory frame. For a general material particle **fixed** in link (i)

$${}^ix = {}^iB\,{}^iX + {}^ir \tag{2.4.2}$$

determines the laboratory coordinates. The kinetic energy of this link is given

$$
\begin{aligned}
K_i &= \frac{1}{2} \int_{link(i)} \| \, {}^i\dot{x} \, \|^2 \, dm \, (\, {}^i x) \\
&= \frac{1}{2} \int_{link(i)} tr \, (\, {}^i\dot{x} \; {}^i\dot{x}^T) \, dm \, (\, {}^i x) \\
&= \frac{1}{2} \int_{link(i)} tr \, ((\, {}^i\dot{x}_1) \, (\, {}^i x^T)) \, dm \, (\, {}^i x) \\
&= \frac{1}{2} \int_{link(i)} tr \, \dot{T}_0^i \, (\, {}^i X_1) \, (\, {}^i X_1^T) \, \dot{T}^{iT}{}_0 \, dm \, (\, {}^i X) \\
&= \frac{1}{2} \, tr \, (\dot{T}_0^i \; \mathbb{J}_I \; \dot{T}_0{}^{iT})
\end{aligned}
\qquad (2.4.3)
$$

medskipwhere,

$$
J_i = \int_{link \; (i)} (\, {}^i X_1) \, ({}^i X^T \; 1) \, dm \, (\, {}^i X)
$$

Remark: If 0 **is** the center of the rigid body, then,

$$
\overset{\circ}{\mathbb{I}}_i \int_{link} {}^i X {}^i X^T \, dm \, (\, {}^i X)
$$

is called the coefficient of inertia. It is easy to verify that for any Ω,

$$
tr(\hat{\Omega} \overset{\circ}{\mathbb{I}}_i \hat{\Omega}) = \Omega \cdot \mathbb{I}_i \, \Omega
$$

where $\mathbb{I}_i$ is the usual moment of inertia.

Returning to (2.4.3), recall from section (1.5) that

$$
T_0^i = T_0^1(q_1) \, T_1^2(q_2) \cdots T_{i-1}^i(q_i) \qquad (2.4.4)
$$

Thus

$$
\mathbb{T}_0^i = \sum_{j=1}^{i} \frac{\partial T_0^i}{\partial q_j}. \qquad (2.4.5)
$$

Hence

$$K_i = \frac{1}{2} \sum_{j=1}^{i} \sum_{k=1}^{i} \dot{q}_j \, \dot{q}_k \, tr \left(\frac{\partial T_0^I}{\partial q_j} \, J_i \, \frac{\partial T_0^i}{\partial q_k} \right) \tag{2.4.4}$$

Suppose to each joint is attached an actuator (that rotates or translates), with stored evergy.

$$K_i^{act} = \frac{1}{2} \, J_i^{act} \, \dot{q}_i^2 \tag{2.4.5}$$

If the joint (i) is an R-pair then J_i^{act} is an axial moment-of- inertia. If it is a P-pair, then J_i^{act} is a mass.

The total kinetic energy of the manipulator

$$K = \sum_{i=1}^{N} (K_i + K_i^{act}) \tag{2.4.6}$$

The potential evergy of the manipulator due to gravity is

$$\begin{aligned} &= \sum v_i + \bar{v} \\ &= -\sum_{i=1}^{N} m_i \, g^T \, T_0^i \, i\tilde{R}_c + \bar{v} \end{aligned} \tag{2.4.7}$$

where $\bar{v}$ = constant corresponding to referance potential energy,

$${}^i\tilde{R}_c = \begin{pmatrix} {}^iR_c \\ 1 \end{pmatrix}$$

where ${}^iR_c$ = center of mass of i^{th} link in i^{th} frame, $g^T = (q_x, \, q_y, \, g_z\sigma)$ is the vector of (uniform) acceleration due to gravity in the laboratory frame, and m_i = mass of ith linked.

The Lagrangian for the manipulator is,

$$L = k - V \tag{2.4.8}$$

The equation of motion are then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \tag{2.4.9}$$

where F_i = joint generalized force / torque using the fact that

$$\frac{\partial T_0^i}{\partial q_p} = 0 \qquad \text{for} \qquad p > i$$

,

and the formulas (2.4.4)-(2.4.9), one can show that

$$F_1 = \sum_{j=1}^{N} D_{ij}\,\ddot{q}_j + J_i^{act}\,\ddot{q}_i + \sum_{j=1}^{N}\sum_{k=1}^{N} D_{ijk}\,\dot{q}_j\,\dot{q}_k + D_i \tag{2.4.10}$$

where,

$$D_{ij} = \sum_{p=max(i,j)}^{N} tr\left(\frac{\partial T_0^p}{\partial q_j}\,J_p\,\left(\frac{\partial T_0^p}{\partial q_I}\right)^T\right)^T \tag{2.4.11}$$

$$D_{ijk} = \sum_{p=max\ (i,\ j,\ k)}^{N} tr\left(\frac{\partial^2 T_0^p}{\partial q_j\,\partial q_k}\,J_p\,\left(\frac{\partial T_0^p}{\partial q_i}\right)\right) \tag{2.4.12}$$

$$D_i = \sum_{p=i}^{N} -m_p\,g^T\,\frac{\partial T_0^p}{\partial q_i}\,p\tilde{R}_c \tag{2.4.13}$$

These are the equations of motion of a manipulator in the absense of contact forces.

The equations are modified if there is gearing at the joints. For a gear pair

$$\begin{array}{cc} \dot{O} & \dot{O}_2 \\ \bigcirc & \bigcirc \\ \uparrow & \uparrow \\ \#\ \text{teeth} = N_1 & N_2 \end{array}$$

"Law of Gearing"

$\frac{\dot{O}_1}{\dot{O}_2} \;=\; \frac{N_2}{N_1}$ (ignoring backlash)

At equilibrium,

$$\frac{\mathrm{TT}_1}{\mathrm{TT}_2} \;=\; \frac{N_1}{N_2}$$

where $\mathrm{TT}_i \;=\;$ torque at gear i.

Let us assume that all joints of the given manipulator have actuators linked to gears with gear ratio $N_0 : 1$, i.e.

$$\dot{\tilde{q}}_i \;=\; \frac{1}{N_0}\,\dot{q}_i, \qquad i \;=\; 1, 2, \cdots, N$$

where $\dot{q}_i \;=\;$ motor/actuator rate and $\dot{\tilde{q}}_i =$ joint rate, for i^{th} joint.

For mechanical advantage with weak actuators, $N_0 \;>>\; 1$.

Now

$$\begin{aligned} L \;=\; & \frac{1}{2}\, J_i^{act}\, N_0^2\, \dot{\tilde{q}}_i^2 \\ & +\; \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{i} \sum_{k=1}^{i} \dot{\tilde{q}}_k\; \dot{\tilde{q}}_i\; tr \left(\frac{\partial T_0^i}{\partial \tilde{q}_k}\; J_i\; \frac{\partial T_0^{i^T}}{\partial \tilde{q}_j} \right) \\ & +\; \sum_{i=1}^{N} m_i\, g^T\, T_0^i\; {}^i\tilde{R}_c \end{aligned}$$

$$
\begin{aligned}
F_i &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \\
&= \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{q}}_i} - \frac{\partial L}{\partial \tilde{q}_i}\frac{\partial \tilde{q}_i}{\partial q_i} \\
&= \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{q}}_i}\frac{1}{N_0} - \frac{\partial L}{\partial \tilde{q}_i}\frac{1}{N_0} \\
\Rightarrow \qquad N_0\, F_i &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{q}}_i} - \frac{\partial L}{\partial \tilde{q}_i}
\end{aligned}
$$

Then

$$
\begin{aligned}
N_0\, F_i = &\sum_{j=1}^{N} D_{ij}\, \ddot{\tilde{q}}_j + J_i^{act}\, N_0^2\, \ddot{\tilde{q}}_i \\
&+ \sum_{j=1}^{N}\sum_{k=1}^{N} D_{ijk}\, \dot{\tilde{q}}_j\, \dot{\tilde{q}}_k + D_i
\end{aligned}
$$

Where $D_{ij},\ D_i,\ D_{ijk}$ are as before except that one substitutes $\tilde{q}_j$ for q_j everywhere in the formulas 2.4.11 - 2.4.13.

It is equivalently,

$$
\begin{aligned}
F = &\sum_{j=1}^{N} \frac{D_{ij}}{N_0^3}\, \ddot{q}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{D_{ijk}}{N_0^3}\, \dot{q}_j \dot{q}_k \\
&+ J_i^{act}\, \ddot{q}_i \\
&+ \frac{D_i}{N_0} \\
= &\ (J_i^{act} + \frac{D_{ii}}{N_0^2})\, \ddot{q}_i + F_{dist}^{i\ a} + F_{dist}^{i\ c} + F_{dist}^{i\ g}
\end{aligned}
$$

where the disterbance terms are

$$
\begin{aligned}
{}^{i}F^{a}_{dist} &= \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{D_{ij}}{N_0^2}\, \ddot{q}_j \\
{}^{i}F^{c}_{dist} &= \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{D_{ijk}}{N_0^3}\, \dot{q}_j\, \dot{q}_k \\
{}^{i}F^{g}_{dist} &= \frac{D_i}{N_0}
\end{aligned}
$$

It is clear that since $J_i^{act} + \frac{D_{ii}}{N_0}$ is the dominant (acceleration) term one can resonably neglect the disturbance terms. This is often done leading to independent joint control techniques.